VECTOR REPRESENTATION FOR ELECTRICAL METERMEN

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ELECTRICAL METERMEN

BY

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First Edition
Second Impression

McGRAW-HILL BOOK COMPANY, Inc.
NEW YORK AND LONDON
1931

PREFACE

Although this volume is intended primarily for metermen and those active in the light and power industry, it is hoped that it will also be found of value to anyone who, for the first time, is undertaking the study of vector representation of alternating voltages and currents.

The apparent scarcity of elementary texts on vectors, coupled with the acknowledged effectiveness of their use in the solution of alternating-current problems, has led the author to undertake the preparation of this text. The presentation has been made as simple as possible in order to meet the needs of those with limited technical training. General conceptions to the contrary, it is the author's earnest opinion that vectors may be presented so that they can be understood and used with reasonable facility by the man in the field.

To accomplish this result it is first necessary to surmount a universal opinion in the average mind, which the very name of vector creates, that vectors are complicated technical tools which no one but a college professor can understand. As a result, Part I of this volume introduces the subject in a non-mathematical, non-technical way. In fact, the name of vector is omitted. Several years' experience in presenting this subject, in this way, to various groups of metermen at meter short courses has proved that the method has merit. Although technical accuracy may be lacking in minor particulars, the benefits gained outweigh the technicalities lost.

After he has studied the presentation given in Part I, it is hoped that the reader will be able to study the more exacting and more technical presentation of Part II with an unbiased mind. Although Part I is self-contained and sufficient for most purposes without Part II, Part II should not be attempted without having mastered Part I, unless one is reasonably familiar with vector theory to start with.

Having mastered the theory of vectors and their manipulation contained in Part I, and preferably in Parts I and II, the reader then should have no trouble in understanding the problems and solutions of Parts III and IV. These problems are selected as being of general application and frequent occurrence. In fact, many of them are actual problems brought to the attention of the author by metermen themselves. Their purpose, however, is not to show a particular solution to a particular problem but to afford a training for the development of the reader's technique in problem attack and analysis. It is only in this way that he can become proficient enough to solve his own problems with assurance and confidence.

In Part IV, a study is made of power factor and power-factor correction. The increasing use of power-factor clauses in rate schedules has resulted in a wider use of power-factor corrective equipment by power customers. As a result, the meter department is frequently called upon to solve a special metering problem in connection with such customers, and it was thought that an appreciation by the meterman of the broader technical and economic problems associated with the specific metering problem would be of considerable benefit. Whenever it was necessary to define the power factor in Parts I, II, and III of this text, an abstract definition was used. The reader was purposely not informed of the basic causal phenomena of the phase displacement between the voltage and current in an alternating-current circuit. To do otherwise would require the use of higher mathematics, which, in a supposedly non-mathematical text, is impossible. As a result, the power factor was defined in terms of meter readings, the constants of the circuit, or the components of phase displacement of the current or voltage. In Part IV, a slightly more thorough treatment is given.

An Appendix has been added which reviews the most frequently used principles of trigonometry. It is by no means a complete treatise on the subject but will be found sufficient for the purpose of this work. It is impossible to exclude trigonometry entirely, but it is not necessary that the reader shall have ever studied it before or that he completely master it. He need know only a few simple relations and how to use a table of trigonometric functions. These will be found sufficiently explained in the Appendix.

In preparing this text, the author is greatly indebted to the helpful suggestions of many of his friends in the meter departments of the Middle West. In particular does he wish to acknowledge the valuable assistance of Profs. C. F. Harding and D. D. Ewing of the School of Electrical Engineering, Purdue University and E. A. LeFever, meter superintendent of the Buffalo General Electric Company and vice chairman of the National

Meter Committee of the National Electric Light Association.

The author will appreciate any suggestions for improvement of the text and any notations of errors which may be discovered.

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PURDUE UNIVERSITY,
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June, 1931.

CONTENTS

PART I

NON-MATHEMATICAL PRESENTATION

VECTOR REPRESENTATION FOR ELECTRICAL METERMEN

CHAPTER I

SYMBOLIC REPRESENTATION OF VOLTAGES AND CURRENTS

1. The Difficulty of Describing Electrical Quantities.

The art of explaining or describing a complicated article accurately and understandingly becomes extremely difficult when that article is an electrical device. This is due, in large measure, to the intangible nature of electricity itself. Electricity is an exceedingly difficult subject for the average human mind to visualize.

Because of this peculiarity of electricity, it has been necessary to develop a method of explaining certain electrical phenomena not generally employed in other fields. The written or spoken work is inadequate to produce the desired result, to say nothing of the length of time involved in its use. Consequently, the vector diagram has been developed as a supplementary aid to the written word. Just as the engineer uses the slide rule to speed up his calculations, and the stenographer, shorthand in order to take dictation more rapidly, so we use the vector diagram to represent electrical quantities. They not only save time, but, when once understood, they are vastly more effective.

Before we take up a study of the vector and the vector diagram, let us develop some logical symbolic method of our own for representing voltages and currents. This will help us better to understand the true significance of the vector representation discussed in Part II.

2. Describing Alternator Loads in Words.

Let us suppose that we wish to explain to another the relations between the voltages and currents in a three-phase alternator when it is connected to (*a*) no load, (*b*) a balanced non-inductive load, (*c*) a balanced load of inductive reactance, (*d*) a balanced load of capacity reactance, and (*e*) an unbalanced load of inductive reactance. To do this in words, we might proceed as follows: A three-phase alternator normally produces three equal sine-wave voltages differing in phase by one-third of a cycle or 120 electrical degrees. Either voltages, or currents, are said to be symmetrical when they have the same wave shape and root-mean-square value and differ in phase from each other by the same angle. For load *a* we have only the three symmetrical voltages. For load *b* we have, in addition to the three symmetrical voltages, three symmetrical currents in phase with their respective voltages. For load *c* we would have three symmetrical voltages and three symmetrical currents. The currents, however, are not in phase with their voltages but lag behind them. The amount of lagging depends upon the power factor. For load *d* the relations are the same as for load *c* except that the currents lead instead of lag behind their voltages. For load *e* we have the three symmetrical voltages but the currents are not of the same magnitude or phase displacement. In general, they lag behind by unequal amounts.

It is obvious that only an engineer thoroughly familiar with the vernacular of the science could completely understand the foregoing. Even then the magnitudes and degrees of the various relationships would be difficult to visualize.

3. Symbolical Representation of Alternator Coils.

Suppose we now attack the description of the voltages and currents produced by a three-phase alternator by representing them symbolically.

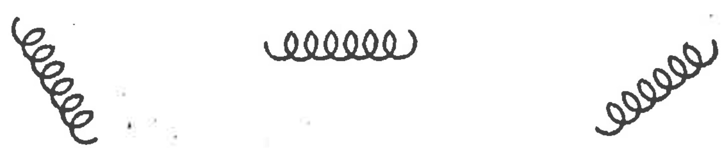


FIG. 1.—Three identical, separate, coils of wire.

Since voltages are always induced in conductors and these conductors are usually in the form of coils, we

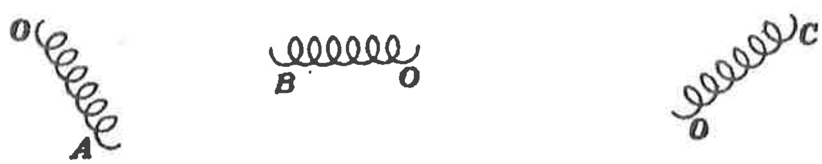


FIG. 2.—The coils of Fig. 1 designated by letters.

take three identical coils of wire and represent them as in Fig. 1. Because these coils are identical, it would be impossible to tell them apart unless we designate them in some suitable manner. This we might do by the letters *OA*, *OB*, and *OC*, as in Fig. 2. We can now refer to a given coil as either coil *OA* or coil *AO*. We could, of course, use numbers in place of the letters. This is sometimes done. However, it would not suffice (as we shall see later) to designate the coils as 1, 2, and 3, for then we could not distinguish between the ends of a given coil.

4. Placing the Coils on an Alternator.

If, now, we place these coils on an alternator, we must do so in some fixed manner. Since there are 360

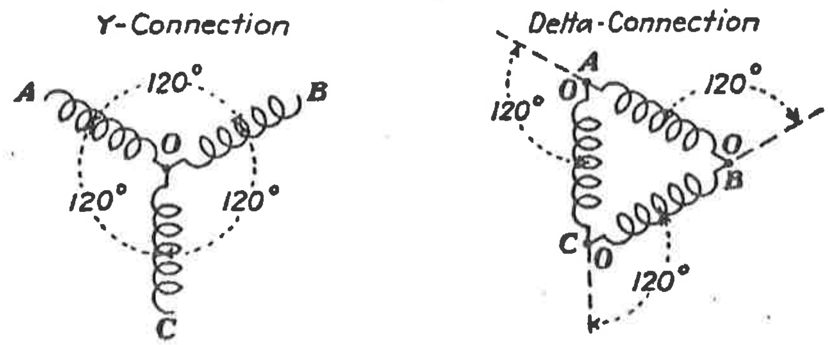


FIG. 3.—Coils of Fig. 1 connected together and placed on an alternator in order *OA*, *OB*, and *OC*.

degrees in a circle, it is natural to do this symmetrically, that is, 120 degrees apart. Also we may connect them in any way we like. Two methods are commonly used as illustrated in Fig. 3. The first is called a Y or star connection and the second, a delta or mesh connection. The Y connection is obtained by connecting similar points of the coils together, and the delta connection by connecting the coils in sequence. In Fig. 3 the coils are connected in the order *OA*, *OB*, and *OC*. We might, of course, have used a different order, as *OA*, *OC*, and *OB* in Fig. 4. We distinguish between these two different arrangements by saying that the phase sequence is different. By phase sequence is meant the order, in a clockwise direction, in which our coils are placed in our diagram. The phase sequence of Fig. 3 is *OA*, *OB*, and *OC*, while that of Fig. 4 is *OA*, *OC*, and *OB*. In either

FIG. 4.—Coils of Fig. 1 connected together and placed on an alternator in order *OA*, *OC*, and *OB*.

event we can still designate a given coil as coil *OA* or coil *AO*.

5. The Use of Lines to Represent Alternator Coils.

As previously mentioned, the object of this method of representation is to save time. One way of saving

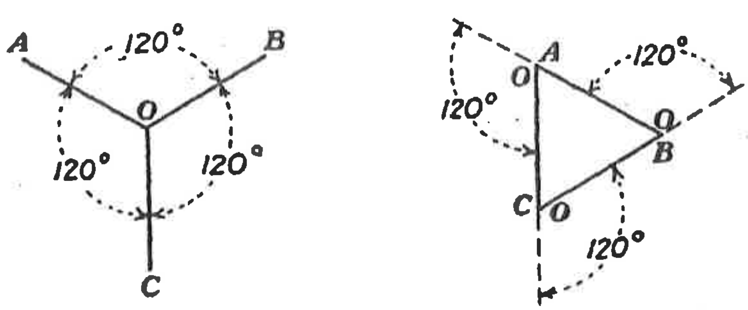


FIG. 5.—The coils of Fig. 3 represented by lines.

time would be to dispense with the curls and represent the coils by straight lines as in Fig. 5. Since the coils are of equal length and fixed in position, the lines must be of equal length and fixed in position. We can still refer to a given coil as line *OA* or as line *AO*.

6. Representing Voltages by Lines.

Now suppose our three coils to be subjected to a special type of uniformly moving, magnetic field such as to induce in each coil simultaneously a continuous electromotive force (e.m.f.) or voltage acting from the

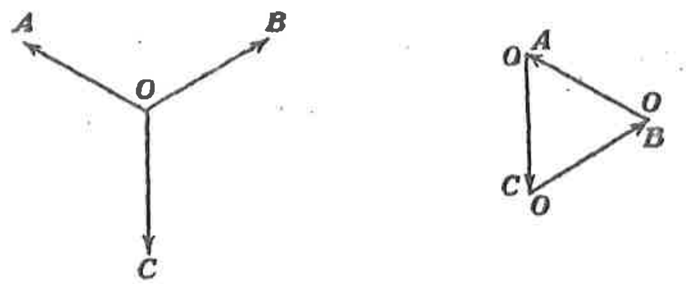


FIG. 6.—Lines representing outwardly induced voltages.

O end out in each coil. In Fig. 5 we have the coils represented by lines, and since the induced voltages are also in the coils, we can easily represent the voltages by the same lines that originally represented coils. In order to show the direction in which the

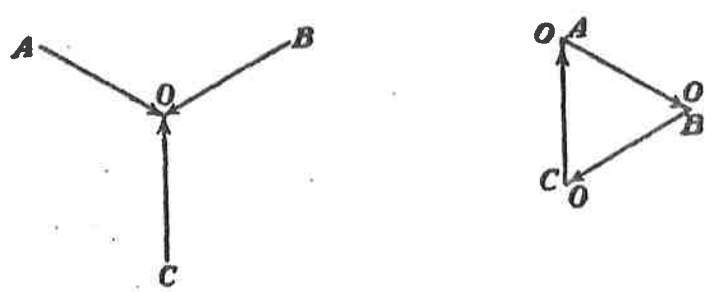


FIG. 7.—Lines representing inwardly induced voltages.

voltages are acting, we place an open arrowhead on the end of each line, as in Fig. 6.

If we were to reverse the direction in which our magnetic field is moving, the voltages induced in our coils would be in the opposite direction. To show this we have only to place the arrowheads on the opposite ends of the lines representing, originally, the coils but now the voltages induced in the coils. The result would be as shown in Fig. 7.

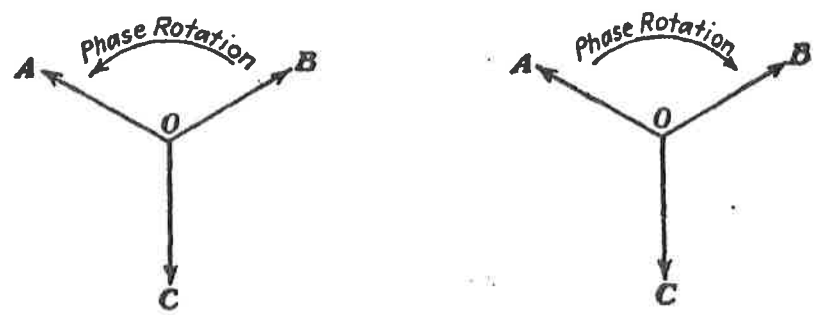
It would not, now, be proper to designate the lines in Fig. 6 by either the notation *OA* or *AO*, etc., because to do so would be to ignore the direction of the induced voltage which these lines represent. We may arbitrarily adopt either method as our conventional representation. Since, however, the former is more generally used, we shall designate a voltage acting from *O* to *A* as E_{OA} and point the arrow toward the second letter in the subscript. In Fig. 7 the proper designation would be E_{AO}, etc.

It will be remembered that the length of the line originally represented the length of the coil it designated. It is a simple step to let this length now represent the magnitude of the induced voltage. Thus, to some arbitrary scale, say 1 inch = 80 volts, we lay off our lines to correspond to the number of volts induced in our coils.

We have now by logical development arrived at a symbol for representing a voltage. This symbol consists of a straight line with an open arrowhead on one end. The direction of the line represents the direction in which the voltage is acting relative to any other line, the arrowhead represents the direction along the line in which the voltage is acting, and the length of the line represents, according to a scale, the magnitude of the voltage. We may also designate the voltage by the letter *E* with a double letter subscript such as *OA*. The order of the letters, as *OA*, in turn designates the direction of the voltage, in this case from *O* to *A*.

7. Representation of the Element of Time and Phase Rotation.

In practice it would be virtually impossible to produce the special type of magnetic field mentioned above. In a commercial machine the field revolves and in so doing sweeps across each coil in turn—not all at once. This introduces a time element which we must not ignore in our symbolic notation. If our field is revolving in a clockwise direction, it will cut across coil *OA* before it does coil *OB* and will cut across



(*a*) Conventional method. (*b*) Special method.

FIG. 8.—Method of designating phase-rotation.

coil *OB* before it does coil *OC* (see Fig. 3). Therefore, the voltage in coil *OA* will reach its maximum value before that in coil *OB* reaches its maximum value, and the voltage in coil *OB* will reach its maximum value before that in coil *OC* reaches its maximum value. The order in which the e.m.fs. in the several coils reach their maximum value in the positive direction is termed phase rotation. We can take this into account in our diagrams, as shown in Fig. 8.

The lines in Fig. 8 represent voltages not coils. The fact that in Fig. 8*a* the phase rotation is counterclockwise means that the voltage E_{OB} reaches its maximum value in the positive direction some time after the voltage E_{OA} has reached its maximum value in the positive direction. Also voltage E_{OC} reaches its maximum value an equal time after E_{OB} has reached its maximum value. On the other hand, the reverse is true in Fig. 8*b*. Here the phase rotation is clockwise. This means that the voltage E_{OC} reaches its maximum value in the positive direction some time after the voltage E_{OA} has reached its maximum value in the same direction. Also the voltage E_{OB} follows E_{OC}. By common consent, counterclockwise phase rotation has been chosen for general use (Fig. 8*a*). As a result, the curved arrow above the diagram is frequently omitted. When it is omitted, counterclockwise phase rotation is always implied.

From what has just been said, it is obvious that the angles between the lines *OA*, *OB*, and *OC* take on a new significance. They now measure time. That is, we measure time in our diagrams by degrees instead of seconds. In the diagrams so far drawn, 120 degrees equals ⅓ cycle. If the frequency is 60 cycles, 1 cycle will occur in 1/60 second and ⅓ cycle will occur in 1/180 second. Thus, 120 degrees actually represents 1/180 second, and a larger or smaller angle would represent, respectively, a greater or a shorter time.

8. Representing Currents by Lines.

Without repeating all that has been said so far, we can conclude, from analogy, that currents can be represented by a line just as voltages were, and referred to as I_{OA}, I_{OB}, or I_{OC}. However, to distinguish lines representing currents from those representing voltages, we put a solid arrowhead on the current line and not an open arrowhead, as in the case of voltages (see Fig. 9).

Our symbolic language is now complete. We have a means of representing a voltage—the straight line. The position of this line represents the position of the voltage, the length of the line the magnitude of the voltage, the arrowhead the direction of the voltage, and the angle the line makes with another voltage line the time-phase displacement, leading or lagging, that the maximum positive value of this voltage leads or lags behind the maximum positive value of the other voltage. We represent a current in exactly the

(*a*) Voltage E_{OA}. (*b*) Current I_{OA}.

FIG. 9.—Symbolic notation for (*a*) voltage and (*b*) current.

same way except that the arrowhead is solid instead of open.

In the case of currents, it will be well to mention, at this point, a fundamental principle of current flow. Current flows in an electrical circuit by virtue of the voltage impressed upon it. This current does not have to and seldom does reach its maximum value at the same instant of time that the voltage which caused it to flow reaches its maximum value. Because of this fact, we may have a phase-time displacement between the current and the voltage in a circuit. This phase-time displacement is represented in our symbolical notation by an angle between the line representing the current and the line representing the voltage.

Whether the current leads or lags behind the voltage depends upon the character of the circuit. Any alternating-current circuit which requires the setting up of a magnetic field will produce a lagging current, and any alternating-current circuit which requires the setting up of an electrostatic field will produce a leading current (see Chap. II, Art. 17). The basic proof of the above two statements requires the use of higher mathematics and consequently will not be given here. This phase displacement of the current with respect to the voltage is the direct cause of the power factor of the circuit. The power factor is explained in Chap. II, Art. 14.

9. Symbolical Description of Alternator Loads.

Suppose, now, that we use our symbols to show the relations of the voltages and currents in our three-phase alternator when connected to the five loads listed in Art. 2. This is done in Fig. 10, using the first method of representation.

Assuming that by this time we are reasonably proficient in our own sign language, Fig. 10 tells us at a glance what took considerable space to say in words. Furthermore, these figures give us a very good idea of the relative magnitudes and phase displacement of the quantities represented. The scales used in Fig. 10 are 1 inch = 200 volts and 1 inch = 20 amperes. It is not necessary that the same scale be used for both voltages and currents.

As an additional drill in the interpretation of our symbols, suppose we reverse the process and analyze each of the diagrams in Fig. 10. The first diagram tells us that we have a machine producing three equal voltages of about 100 volts, since the lines are equal in length and about ½ inch long and have open arrows on their ends. The voltages are displaced from each other by 120 degrees, and since our conventional phase rotation is counterclockwise, this means that voltage E_{OB} reaches its positive maximum 120 degrees (1⁄180 second at 60 cycles) after E_{OA} has

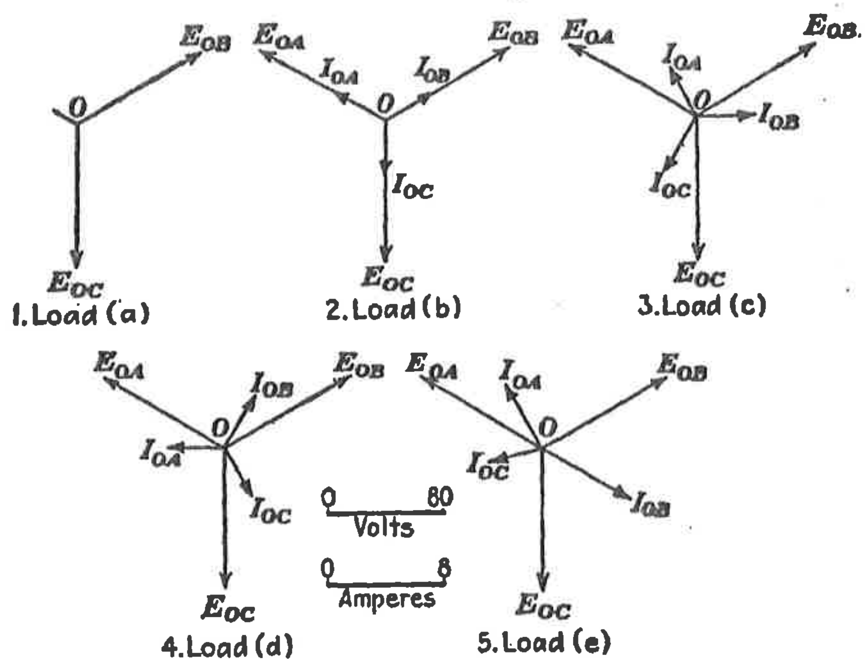


Fig. 10.—Symbolical representation of five types of loads for a Y-connected three-phase alternator.

reached its positive maximum. In like manner, E_{OC} reaches its positive maximum 120 degrees after E_{OB}. The second diagram tells us that the first machine now has a load, since there are current lines shown, *i.e.*, lines with solid arrowheads. This load is balanced and equal to about 4 amperes, since the current lines are equal and about 3⁄16 inch long. The load is also non-inductive, *i.e.*, a unity power-factor load (see Chap. II, Art. 17), since the current lines coincide with the voltage lines. There is, therefore, no time lag between the currents and the voltages. They both reach their positive maximum at the same instant. The third diagram shows the same condition as the second except that this load has inductive reactance. We know it is inductive reactance because the currents lag behind their voltages (see Chap. II, Art. 17). They do not reach their positive maximum until about 30 degrees after their voltages have reached

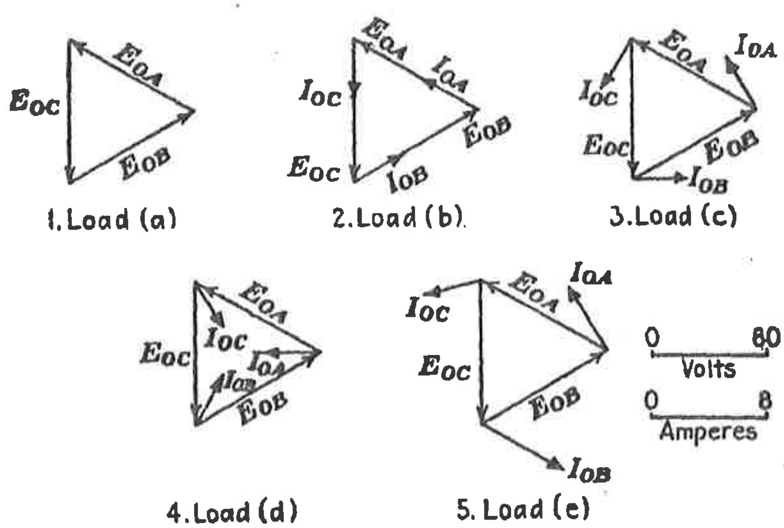


Fig. 11.—Symbolical representation of five types of loads for a delta-connected three-phase alternator.

their maximum. This would correspond to $\frac{1}{4} \times \frac{1}{180} = \frac{1}{720}$ second on a 60-cycle system and to a power factor of about 86.5 per cent (see Chap. II, Art. 15). The fourth diagram is identical with the third except that the load contains capacity reactance drawing a leading current (see Chap. II, Art. 17). We know it is leading because the current lines are drawn about 30 degrees ahead of their voltages and therefore reach their maximum values about $\frac{1}{720}$ second ahead of their voltages. The last diagram shows an unbalanced load. We know it is unbalanced because the current lines are not equal. The current I_{OB} is the largest (about 7 amperes) and lags about 60 degrees behind its voltage. I_{OA} is about 5 amperes and lags 30 degrees behind its voltage. I_{OC} is about 4 amperes and lags 65 degrees behind its voltage.

In Fig. 11 the same diagrams are repeated using the second method of representation. The reader should analyze these figures in a manner similar to the above.

10. Problems

1. Represent a voltage of 120 volts when 1 inch = 80 volts.

2. Represent a current of 10 amperes when 1 inch = 8 amperes.

3. If 120 degrees = $\frac{1}{180}$ second, how many seconds will 60 degrees equal?

4. Show a current of 10 amperes which lags 30 degrees behind a voltage of 120 volts.

5. If a certain 60-cycle current I_{OA} lags behind a certain voltage E_{OA} by $\frac{1}{480}$ second, what is the angle of lag?

6. Represent three voltages E_{OA}, E_{OB}, and E_{OC} differing in phase by 120 degrees so that E_{OB} lags behind E_{OC} and E_{OA} lags behind E_{OB}.

7. Show a current of 8 amperes leading a voltage of 100 volts by 45 degrees.

8. Show a voltage E_{OA} = 120 volts, a voltage E_{OB} = 80 volts, a voltage E_{OC} = 100 volts, and a current I_{OA} = 10 amperes, when E_{OA} leads E_{OB} by $\frac{1}{720}$ second, E_{OC} lags behind E_{OB} by $\frac{1}{360}$ second, and I_{OA} is in phase with E_{OA}.

9. Represent a balanced three-phase load for a Y-connected alternator of 10 amperes. This load contains inductive reactance and the currents lag behind the voltages by 30 degrees. The voltage of the alternator is 120 volts.

10. A certain 120-volt, delta-connected alternator has an unbalanced load consisting of $I_{OA} = 9$ amperes and $I_{OB} = I_{OC} = 5$ amperes. I_{OA} lags behind E_{OA} by 30 degrees, I_{OB} lags behind E_{OB} by 60 degrees, and I_{OC} is in phase with E_{OC}. The phase rotation is E_{OA}, E_{OB}, and E_{OC}. Draw a complete symbolical representation of these voltages and currents.

CHAPTER II

MANIPULATION OF OUR SYMBOLS

In Chap. I we developed a system of symbols for representing alternating-current quantities. In this chapter we shall see how it is possible to add, subtract, and multiply these symbols. This we must be able to do, since it is often necessary to add two or more currents or voltages or find their difference. And in the case of power measurements we must be able to multiply a voltage and a current.

11. Addition of Two or More Symbols.

We have implied in Chap. I that our symbols represent the maximum value of the voltages and currents they stand for. This, however, is not essential. They may represent any value, provided this value bears a fixed relation to the maximum value. In practice, it is convenient to use what is called the effective value of alternating currents and voltages, because it is this value that a meter records. The effective value of an alternating current is that value which will produce the same heating effect as the same value of direct current. Thus 10 amperes alternating current (effective value) or 10 amperes direct current will produce the same heating effect. The actual relation between the maximum and effective value is of no particular consequence so far as the present discussion is concerned.[1] In this chapter we shall assume that we are dealing with the effective value—that value actually recorded by a meter.

To add two or more voltages or currents symbolically, we make use of a graphical construction. The exact process used depends upon whether the quantities are in phase or out of phase. When two or more quantities are in phase with each other their symbolic representations are parallel lines. These lines may be superimposed upon each other, as E_{OA}

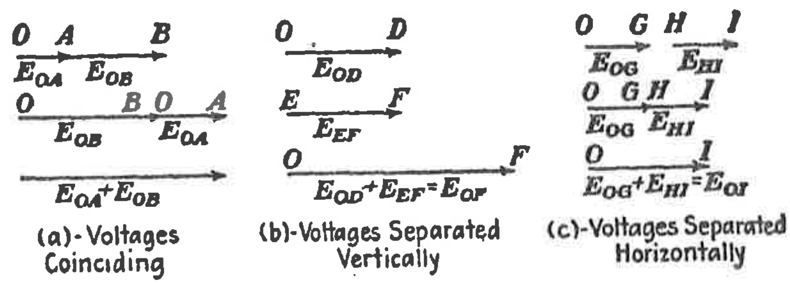


Fig. 12.—Addition of in-phase quantities.

and E_{OB} (Fig. 12*a*); separated vertically, as E_{OD} and E_{EF} (Fig. 12*b*); or separated horizontally, as E_{OG} and E_{HI} (Fig. 12*c*). In all three cases, their sum is found by adding their lengths together. This produces another line, parallel to those being added and equal in length to the sum of their individual lengths. This process is illustrated in Fig. 12. Note how the subscripts are added as well as the lines.

When two or more quantities that are not in phase are to be added, we may employ either one of two methods: (1) Complete the parallelogram or (2) complete the triangle. Since both methods have certain advantages under certain conditions, they will

[1] See Chap. IV, Art. 23.

both be illustrated. To illustrate method 1, suppose we wish to add two voltages E_{OA} and E_{OB}, differing in phase by the angle θ (theta)[1] (Fig. 13*a*). We draw a line parallel and equal to *OB* from *A*, and another parallel and equal to *OA* from *B*. These lines intersect in *D*. A line drawn from *O* to *D* will represent the sum of the two voltages. Its length is the magnitude of the new voltage; and its position relative to the other two, its phase position with respect to them. To illustrate method 2, suppose we wish to add two voltages E_{OA} and E_{BC} which

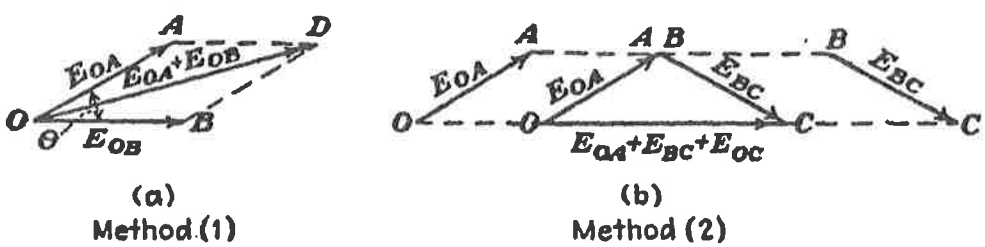


Fig. 13.—Addition of out-of-phase quantities.

are out of phase (Fig. 13*b*). To do this we move either voltage parallel to itself and fasten it on to the arrow end of the other. A line drawn from the beginning of the first to the arrow end of the second will then represent their sum in magnitude and phase position. A little study will show that these two methods are geometrically the same. For instance, in Fig. 13*a*, line *BD* is parallel and equal to E_{OA} and therefore is the same thing as moving E_{OA} parallel to itself and attaching it to the end of E_{OB}. The method to use in any given case depends simply

[1] By custom and convention, the Greek alphabet is used to represent certain mathematical constants and symbols and among them angles. For a complete list of the Greek letters and the quantities they customarily represent, see the "Handbook of Electrical Metermen," p. 1153, 1923.

upon which is the easier. Figure 14 illustrates the addition of a number of different voltages, using both methods. Figure 14*d* and *e* illustrates the sum of more than two symbols.

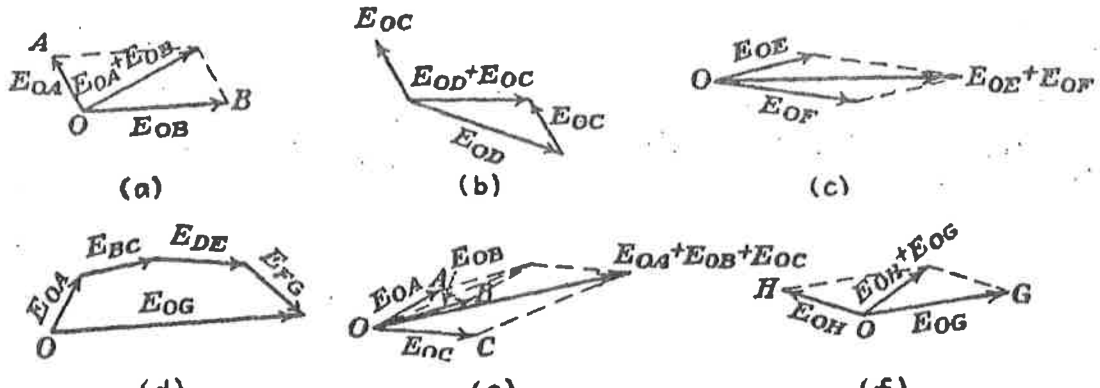


FIG. 14.—Examples of the addition of voltages.

12. The Difference of Two Symbols.

To find the difference of two voltages or currents symbolically, we proceed in a manner similar to addition. If the quantities are in phase, we take the difference of their lengths instead of the sum. Thus to

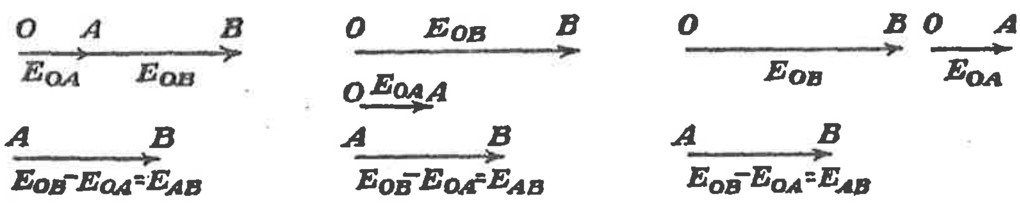


FIG. 15.—Difference of in-phase quantities.

subtract E_{OA} from E_{OB}, we subtract the length of E_{OA} from E_{OB} and the result is

$$E_{OB} - E_{OA} = E_{AB}.$$

This process is illustrated in Fig. 15.

To subtract two quantities that are not in phase, we can, as in the case of addition, employ either of two methods: (1) completing the parallelogram and (2) completing the triangle. To employ method 1 reverse the quantity to be subtracted and add it to the other. Thus, in Fig. 16*a*, to find $E_{OB} - E_{OA}$, we reverse E_{OA} and then add it to E_{OB} by completing the parallelogram. In Fig. 16*b* the triangle method was used. A line connecting the arrowheads from *A* to *B* will be equal to $E_{OB} - E_{OA} = E_{AB}$. This is so because, if we now apply method 1, we obtain the line *OD*. However, line *OD* is parallel and equal to line *AB*. Therefore either *OD* or *AB* will give the

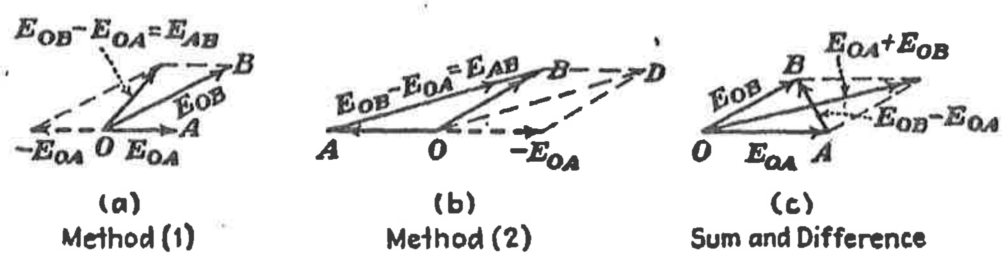


FIG. 16.—Difference of out-of-phase quantities.

desired result. Figure 16*c* shows both the sum and difference of two quantities in the same diagram.

The manipulation of the subscripts should be carefully noted. In Fig. 16*b*, when we subtracted the voltage E_{OA} from E_{OB}, we actually obtained the voltage from *A* to *B* or E_{AB}. The arrowhead of the resultant is, therefore, on the *B* end. Had we wished to subtract E_{OB} from E_{OA}, we should have obtained the voltage from *B* to *A* or E_{BA}. This would be the same line, but the arrowhead would now be on the *A* end.

Figure 17 illustrates the process of subtracting two or more symbols from one. One way of doing this is to add all those to be subtracted and then subtract their sum from the remaining symbol. In Fig. 17*a* method 2 is used; and in Fig. 17*b*, method 1. In the latter method the vectors may be added before or after reversal, as illustrated.

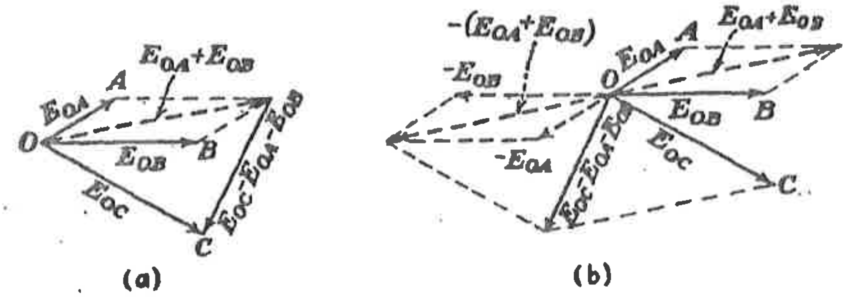


Fig. 17.—The difference of more than two voltages.

13. Multiplication of Two Symbols.

To multiply two symbols, such as a current and a voltage for power calculations, it is necessary to employ a property of our symbols, which will now be explained. It is possible to divide any symbol into the sum of two component symbols at right angles

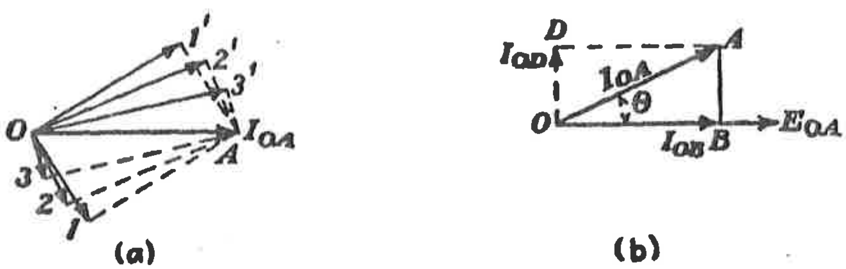


Fig. 18.—Components of a symbol.

to each other. A symbol can have an infinite number of such pairs of components that are at right angles to each other. This is illustrated in Fig. 18*a*. It will be noticed that the sum of each set of components equals the symbol I_{OA}. Thus $01 + 01' = 02 + 02' = 03 + 03' = I_{OA}$. It is obvious that there could be any number of such pairs of components drawn to a given symbol.

In order to multiply one symbol by another, we first find that component of the second that is in phase with the first, namely, that component which would fall on top or coincide with the first symbol, and multiply this in-phase component by the first symbol. The easiest way to determine the in-phase component of the second symbol is to draw a line from the arrow end of this symbol, which will intersect the other symbol at right angles (90 degrees). Consider the voltage E_{OA} and the current I_{OA}, Fig. 18*b*. To find the in-phase component of I_{OA} we draw line *AB* so that it makes a right angle (90 degrees) with E_{OA}. I_{OB} is then the in-phase component. The other component I_{OD}, called the quadrature component, is ignored in power measurements. However, as we shall see later, it is the quadrature component I_{OD} that is used in reactive volt-ampere measurements.

14. Symbolical Determination of Single-phase Power and Power Factor.

To obtain the power in any single-phase circuit we multiply the voltage by the in-phase component of the current. Thus, in the example of Fig. 18*b*, the power is $E_{OA} \times I_{OB}$ and not $E_{OA} \times I_{OA}$. As an example, suppose that a voltmeter in a certain circuit reads 120 volts, and that an ammeter in the same circuit reads 10 amperes. Assuming that the phase angle is as shown in Fig. 18*b*, we lay off E_{OA} ¾ inch long and I_{OA} ⅝ inch long. If now we scale off I_{OB} we find it to be 9⁄16 inch long or equal to 9 amperes. Therefore the power consumed is $9 \times 120 = 1{,}080$ watts and not $10 \times 120 = 1{,}200$ watts as might be expected.

The ratio of the actual power 1,080 to the apparent power 1,200 is called the power factor.[1] Thus, in this example,

$$\text{Power factor} = \frac{\text{actual power}}{\text{apparent power}} = \frac{E_{OA} \times I_{OB}}{E_{OA} \times I_{OA}} = \frac{1{,}080}{1{,}200} = 0.9 \text{ or } 90 \text{ per cent.} \quad (1)$$

Clearing fractions,

Actual power = apparent power × power factor.

It follows that for single-phase circuits,

$$P = EI \times \text{power factor}, \quad (2)$$

where

P = power in watts,
E = e.m.f. in volts,

and

I = current in amperes.

In the special case when the voltage and the current are in phase, the power is obtained by multiplying the voltage and current together directly. Thus, in the above example, if E_{OA} and I_{OA} had been in phase, the power would have been $120 \times 10 = 1{,}200$ watts. In this case the apparent power and actual power are equal and the power factor is one or unity.

To measure the power factor it is necessary to record the actual power directly. To do this we use a wattmeter. Then the wattmeter reading, divided by the product of the voltmeter and ammeter reading,

[1] See Part IV, Chap. X, Art. 47.

gives, as we have just seen, the power factor. Thus, experimentally,

$$\text{Power factor} = \frac{\text{wattmeter reading}}{\text{voltmeter reading} \times \text{ammeter reading}}. \quad (3)$$

The power factor is a number which may vary from one to zero. It is never greater than one and only in one special case when the current and voltage are in phase is it equal to one.

15. Determination of the Phase Angle.

The actual value of the phase angle may be determined from the power factor. In the single-phase circuit the power factor has another significance. It is also equal to the cosine of the angle between the voltage and the current. The cosine (abbreviated cos) is one of the common trigonometric ratios.[1] Thus

$$\cos \theta = \text{power factor} = \frac{\text{wattmeter reading}}{\text{voltmeter reading} \times \text{ammeter reading}} = \frac{1{,}080}{120 \times 10} = 0.9. \quad (4)$$

Substituting cos θ (cosine theta) for the power factor Eq. (2), we get

$$P = EI \cos \theta. \quad (5)$$

This is the power equation for the single-phase circuit. With the value of the cosine as found by Eq. (4) and a table of trigonometric functions, we can determine the value of θ of Fig. 18*b*. In this case

[1] See Appendix.

the angle whose cosine is 0.9 is 25 degrees 48 minutes or approximately 26 degrees.

It is impossible to sidestep the use of simple trigonometric functions. However, it is not essential that the reader have a complete understanding of them. The abbreviated elementary treatment contained in Appendix I will be found sufficient for anything contained in this text.

The above process may be reversed. That is, if we know the phase angle in degrees, we can obtain the power factor from the table of trigonometric functions. Since the power factor equals the cosine of the phase angle, we look up the cosine corresponding to the phase angle and that is also the power factor. For example, if a certain current lags behind its voltage by 30 degrees, the power factor is 0.866 or 86.6 per cent ($\cos \theta = \cos 30$ deg. $= 0.866$).

16. Mathematical Determination of Single-phase Power.

In Art. 14 we obtained the value oı the in-phase component of the current, graphically, by scaling it from the diagram. We now have a means of doing this mathematically. We have just seen that the power factor, which equals also the cosine of the phase angle, was equal to 0.9. If we multiply the current $I_{OA} = 10$ amperes by 0.9 we get 9 amperes which is equal to I_{OB}, the in-phase component of I_{OA}. Therefore to obtain the in-phase component of any current, multiply it by the cosine of the angle by which it leads or lags behind the voltage. Since the power is the product of the voltage and the in-phase component of the current, it may be found as follows:

$$\text{In-phase component of } I_{OA} = I_{OB} = I_{OA} \times \cos \Theta = 10 \times 0.9 = 9.0.$$

$$\text{Power} = P = E_{OA} \times I_{OB} = E_{OA} \times I_{OA} \times \cos \Theta = 120 \times 10 \times 0.9 = 1{,}080,$$

or, using Eq. (5),

$$P = EI \cos \Theta = 120 \times 10 \times 0.9 = 1{,}080.$$

17. Rules of Phase Relation.

We have just seen how we obtain, by the use of instruments, the value of the phase angle. However, there is no way of telling from these instruments alone whether this is a leading or a lagging angle. As previously mentioned, whether the current leads or lags, the voltage depends upon the character of the load. Three rules will now be given for determining the phase relation:

a. The current is in phase with the voltage when the load is a pure ohmic resistance, or non-inductive. Examples of such loads are: (1) the incandescent lamp, (2) a carbon-pile rheostat, and (3) some specially wound slide-wire rheostats.

b. The current lags behind the voltage when the load has inductive reactance. Examples of such loads are: (1) induction motors, (2) ordinary coils, and (3) circuits containing iron cores. When this inductive reactance is pure, namely, no resistance present, the current lags behind the voltage by 90 degrees.

c. The current leads the voltage when the load has capacity reactance. Examples of such loads are: (1) static condensers, (2) synchronous condensers, and (3) capacity is also present in some long transmission lines. When the capacity reactance is pure, that is, no resistance is present, the current leads by 90 degrees.

Besides these rules, five others will be found convenient in drawing symbolic diagrams such as we have been studying. These rules are as follows:

d. If the circuit to be represented is a series circuit, it will usually be best to use the current as a reference symbol, because the current is common to all parts of the circuit.

e. If the current to be represented is a parallel circuit, it will be best to use the voltage as a reference, because, here, the voltage is common to all parts of the circuit.

f. Quantities that are in phase with each other add algebraically.

g. Quantities that are out of phase with each other add symbolically.

h. Power adds algebraically always.

The following examples will illustrate the use of these rules:

1. A series circuit of 10 amperes consists of an inductive reactance, a condenser, and a non-inductive resistance. The drop across the reactance is 50 volts at a power factor of 0.8. The drop across the condenser is 30 volts at a power factor of 0.5, and the

drop across the resistance is 20 volts. Find the line voltage and the power factor of the system.

Since this is a series circuit, we use the current as a reference (Rule *d*) and lay off any line I_O (Fig. 19). It need not be to scale. We obtain, from a table of trigonometric functions,[1] the angle whose cosine is 0.8. This we find to be about 37 degrees. From rule *b* we see that the current must lag behind the voltage. Therefore the voltage symbol must be drawn ahead of our current reference as E_{OA}. In the same way we find that the angle whose cosine is 0.5 is 60 degrees,

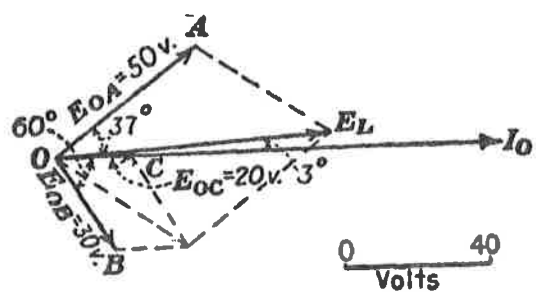


Fig. 19.—Symbolical solution of example 1.

and from rule *c* we find that the current leads the voltage. Therefore this voltage symbol is drawn lagging from our reference I_O as E_{OB}. The drop across the resistance is in phase with the current or E_{OC} (Rule *a*). To find the line voltage we add the voltages E_{OA}, E_{OB}, and E_{OC} according to Rule *g*, obtaining E_L. Scaling E_L we find it to be about 75 volts. To find the power factor we measure the angle between E_L and I_O with a protractor, obtaining about 3 degrees. From our trigonometric tables, we find that the cosine of 3 degrees is about 0.999. Therefore the power factor is 99.9 per cent lagging.

[1] See Appendix.

2. Three loads are connected in parallel to 240-volt mains. Load A is an inductive reactance drawing 20 amperes at 70.7 per cent power factor, load B is a condenser drawing 10 amperes at 10 per cent power factor, and load C is a non-inductive resistance drawing 5 amperes. Find the total current and the power factor of the system.

Since this is a parallel circuit, we use the voltage as a reference (Rule e) and lay off E_o (Fig. 20). In this

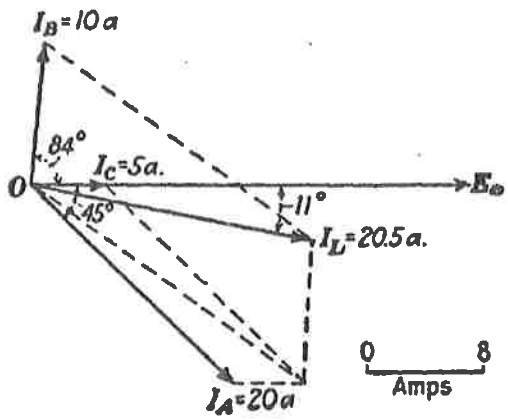


FIG. 20.—Symbolical solution of example 2.

case E_o need not be to scale. Applying Rule b, we lay off I_A lagging behind E_o by the angle whose cosine is 0.707 = 45 degrees. Applying Rule c, we lay off I_B leading E_o by the angle whose cosine is 0.1 = 84 degrees approximately. Applying Rule a we lay off I_C in phase with E_o. To determine the total current, we add I_A, I_B, and I_C according to Rule g, obtaining I_L = 20.5 amperes. Since I_L makes an angle of 11 degrees with E_o, the power factor is about 98 per cent lagging.

3. In the circuit shown in Fig. 21 the meters read as indicated. Prove that the line meters are reading

correctly. Using Eq. 3 and a set of trigonometric functions, we obtain the phase angles as follows:

$$\text{Power factor of load } A = \frac{P}{EI} = \frac{500}{10 \times 100} = 0.5,$$

angle of lag = 60°.

$$\text{Power factor of load } B = \frac{P}{EI} = \frac{433}{5 \times 100} = 0.866,$$

angle of lag = 30°.

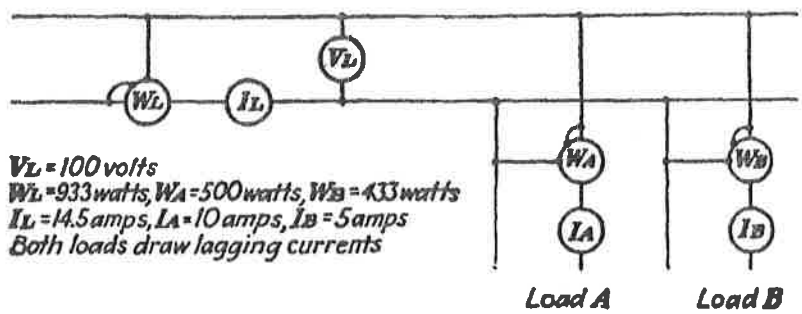


FIG. 21.—Circuit for example 3.

This is a parallel circuit, so we draw our diagram according to Rule *e* and Rule *b*, as in Fig. 22. We obtain the line current I_L by adding I_A and I_B, according to Rule *g*. I_L scales 14.5 amperes. This agrees with the reading of the line ammeter. I_{OC}, the in-phase component of I_L, scales 9.33 amperes. The line power is then

$$E_o \times I_{OC} = 100 \times 9.33 = 933 \text{ watts},$$

which agrees with the reading of the line wattmeter. The angle θ scales 50 degrees, and its cosine, from the tables, is 0.643. Using Rule *h*, we can obtain the line power by adding the load powers.

$$\text{Line power} = 500 + 433 = 933,$$

and the line power factor is equal to

$$\frac{P}{EI} = \frac{933}{100 \times 14.5} = 0.643,$$

which also agrees with the previous value. This problem is not intended to illustrate a method for calibrating meters. It is used to show the application of the rules given above and the interrelations of the various quantities in such a circuit.

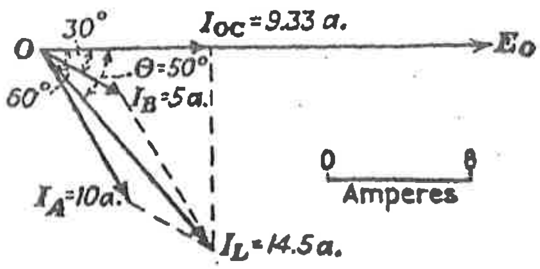


FIG. 22.—Symbolical solution of example 3.

18. Problems

1. Given $E_{OA} = 100$ volts, $E_{OB} = 50$ volts, and the angle between them equal to 45 degrees. Find their sum in two ways.

2. A current of 10 amperes and a current of 8 amperes are 60 degrees out of phase. What is their sum?

3. A voltage of 120 volts is 30 degrees ahead of another voltage of 80 volts, which in turn is 60 degrees ahead of another voltage of 60 volts. What is the sum of these three voltages?

4. A voltage of 80 volts is to be subtracted from a voltage of 120 volts. If the phase angle between them is 60 degrees, what is their difference?

5. Three voltages E_{OA}, E_{OB}, and E_{OC} are all 100 volts. E_{OA} lags behind E_{OB} by 30 degrees and E_{OC} lags behind E_{OB} by 30 degrees. Find $E_{OA} - E_{OB} - E_{OC}$ in two ways.

6. Two currents $I_{OA} = 10$ amperes and $I_{OB} = 5$ amperes differ in phase by 60 degrees. Find their sum and difference.

7. Given a current of 15 amperes, find two components at right angles, one of which leads the given current by 30 degrees.

8. In Prob. 7, find two components that are equal. What is the phase difference between these components and the current?

9. In a certain circuit a current of 10 amperes lags behind the voltage by 60 degrees. What is the in-phase component of this current?

10. The load on a certain system consists of a current of 10 amperes lagging behind the applied voltage by 45 degrees and a current of 5 amperes lagging behind the applied voltage by 30 degrees. (*a*) What is the main line current? (*b*) What is its phase angle? (*c*) What is the in-phase component of this current?

11. Find the total power in Prob. 10.

12. In a certain single-phase circuit the voltmeter reads 120 volts; the ammeter, 20 amperes; and the wattmeter, 1,200 watts. Determine the power factor, phase angle, and draw the complete symbolical diagram, if this load is (*a*) an induction motor, (*b*) a static condenser.

13. In Prob. 12, what is the in-phase component of the current in each case?

14. A certain load has a power factor of 0.866 lagging. If the current is 6 amperes and voltage 110 volts, what is (*a*) the power, (*b*) the phase angle, and (*c*) the in-phase component of the current? Draw the symbolical diagram.

15. A certain induction motor draws a lagging current of 25 amperes at 0.5 power factor from 240-volt mains. A synchronous motor attached to the same mains draws a current of 40 amperes at a power factor of 0.866 leading. What is the line current; the power factor?

16. In the circuit of Fig. 23, the load meters read as shown. What should the line meters read?

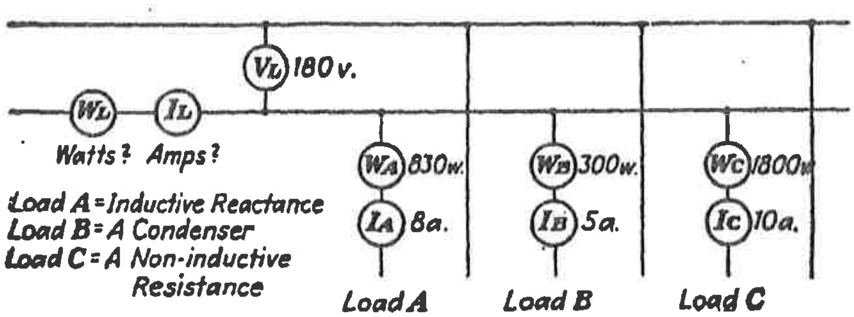


FIG. 23.—Circuit for Prob. 16.

PART II

MATHEMATICAL DEVELOPMENT OF THE VECTOR AND THE VECTOR DIAGRAM

CHAPTER III

THE VECTOR

The reader has probably surmised long before this that the symbolical representation discussed in Part I is nothing more or less than vector representation. The symbols were not called vectors, because the very name of vector invariably creates in the beginner's mind an impression that the vector is a complicated mathematical tool impossible to understand or to apply unless one has had a thorough mathematical training. The purpose of the presentation given in Part I was to evade this difficulty. As far as the practical application of the vector is concerned, he will find it possible to understand the solutions given in Parts III and IV of this text without Part II. However, for those who wish to obtain a keener appreciation of the vector and vector theory, the more exacting development given in Part II will be of great assistance. Care was taken in Part I to use a minimum of mathematics, technical terms, and electrical theory. In Part II the reader will find it to his advantage to review the subject of elementary trigonometry as outlined in Appendix I before proceeding.

19. Scalars and Vectors.

Quantities generally may be divided into two groups called scalars and vectors.

A scaler quantity is one which is completely determined by its magnitude alone. This magnitude, however, may be either positive or negative. Examples of electrical scalars are power, continuous currents and voltages, and energy. Non-electrical scalars are dollars, mass, temperature, gallons, and time. Scalar quantities are added algebraically.

Thus

$$10 \text{ gal.} + 5 \text{ gal.} = 15 \text{ gal.}$$

and

$$\$10 - \$5 = \$5.$$

A vector quantity is one that requires for its complete description both magnitude and direction. Such a quantity is best represented by a straight line the length of which, to some arbitrarily chosen scale, represents its magnitude and the direction of which is parallel to the direction of the quantity being represented. Examples of non-electrical vector quantities are forces and velocities; and of electrical vector quantities alternating currents and voltages. Vector quantities cannot be added algebraically but must be added vectorially. The process of adding or subtracting vectorially is identical to that employed in the case of our symbols of Part I. In Part I to obtain the magnitude of a resultant symbol or vector, we scaled its length. We shall now develop a process for calculating this length.

20. Vector Addition and Subtraction.

The direction of a vector quantity is usually defined with respect to two arbitrarily chosen lines at right angles to each other and in the plane of the vector quantity. These lines are called axes of reference. Thus in Fig. 24a, lines XOX' and YOY' are a set of reference axes. The line XOX' is called the horizontal axis and the YOY' the vertical axis. Quantities measured along OX and OY are positive, while quantities measured along OX' and OY' are negative. These axes divide the plane in which they are located into four quadrants numbered as shown (see also Appendix I). Suppose a body is

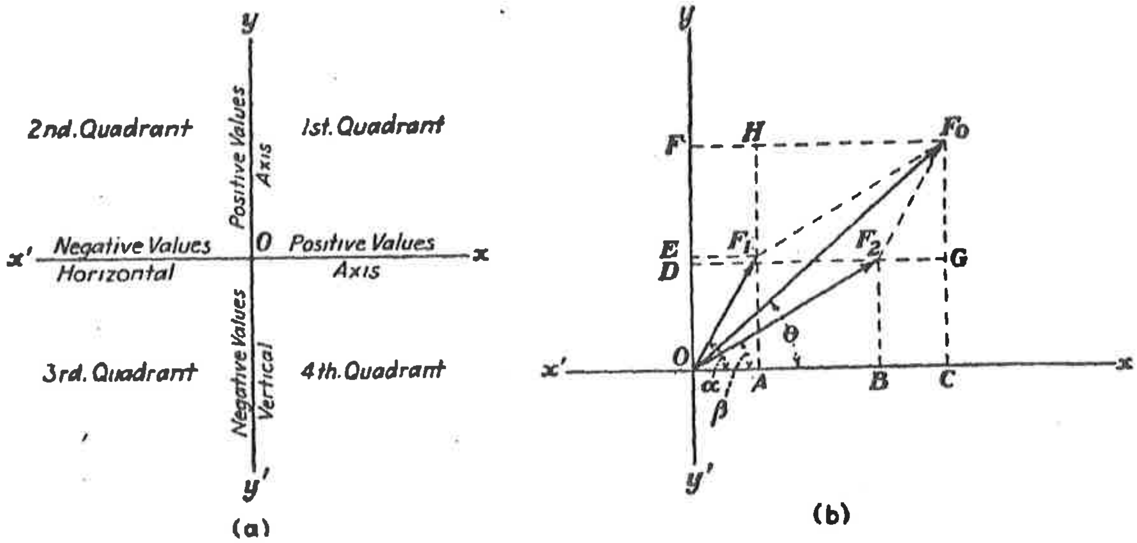


FIG. 24.

acted upon by two forces F_1 and F_2 which are not in the same direction. The resultant force F_0 upon the body will be the vector sum of the forces F_1 and F_2. Let the two forces be defined with respect to a set of axes as in Fig. 24b. F_1 makes an angle α (alpha) with the horizontal axis, and F_2 angle β (beta). The resultant of these two forces F_0 is found by completing the parallelogram (see Chap. II, Art. 11). This resultant makes an angle θ (theta) with the horizontal axis. Its magnitude may be determined by scaling off the vector OF_0 as was done in Part I. However, it is possible to calculate this value from the geometry of the figure. The latter process will, in general, be more accurate. As explained in Chap. II, Art. 13, it is possible to resolve a vector into two components at right angles to each other. This is done in the present case, where OA is the horizontal component and OE the vertical component of F_1. Both F_2 and F_0 are also resolved into a horizontal and vertical component. We see from the geometry of the figure that

$$F_2F_0 = OF_1,$$

opposite sides of a parallelogram.

Therefore:

$$OA = F_2G = BC.$$

Hence:

$$OA + OB = BC + OB = OC.$$

Substituting for OA its equal BC.

Thus:

Horizontal component (H-comp.) of F_1 + H-comp. F_2 = H-comp. F_0.

Also:

$$F_1F_0 = OF_2$$

Opposite sides of a parallelogram.

Therefore:

$$OD = F_1H = EF.$$

Hence:

$$OD + OE = EF + OE = OF.$$

Substituting for OD its equal EF.

Thus:

Vertical component (V-comp.) of F_2 + V-comp. F_1 = V-comp. F_0.

Finally:

$$OF_0 = \sqrt{\overline{OC}^2 + \overline{OF}^2}$$[1]

Now

$$\begin{aligned}
\text{H-comp. } F_1 &= OF_1 \cos \alpha = OA,^1\\
\text{H-comp. } F_2 &= OF_2 \cos \beta = OB,\\
\text{V-comp. } F_1 &= OF_1 \sin \alpha = OE,\\
\text{V-comp. } F_2 &= OF_2 \sin \beta = OD,\\
\text{H-comp. } F_0 &= OF_0 \cos \Theta = OA + OB = OC,\\
\text{V-comp. } F_0 &= OF_0 \sin \Theta = OE + OD = OF.
\end{aligned}$$

Since

$$OF_0 = \sqrt{\overline{OC}^2 + \overline{OF}^2}$$

$$F_0 = \sqrt{(F_0 \cos \Theta)^2 + (F_0 \sin \Theta)^2} = \sqrt{\left(\sum \text{H-comp.}\right)^2 + \left(\sum \text{V-comp.}\right)^2}, \quad (6)$$

$$\tan \Theta = \frac{OF}{OC} = \frac{F_0 \sin \Theta}{F_0 \cos \Theta} = \frac{\Sigma \text{V-comps.}}{\Sigma \text{H-comps.}}. \quad (7)$$

The Greek letter Σ (sigma) is the symbol for algebraic summation, pronounced summation horizontal components. Equations 6 and 7 define the resultant OF_0.

As a numerical example, suppose that $F_1 = 11$ pounds and that $\alpha = 60$ degrees; also that $F_2 = 18$ pounds and that β is 30 degrees. From a table of trigonometric functions, we find that $\cos \alpha = 0.5$, $\sin \alpha = 0.866$, $\cos \beta = 0.866$, and $\sin \beta = 0.5$.

Therefore:

$$\begin{aligned}
\text{H-comp. } OF_1 &= OF_1 \times \cos \alpha = 11 \times 0.5 &&= 5.5\\
\text{H-comp. } OF_2 &= OF_2 \times \cos \beta = 18 \times 0.866 &&= \underline{15.6}\\
\text{H-comp. } OF_0 &= OF_0 \times \cos \Theta = \text{the sum} &&= 21.1
\end{aligned}$$

[1] See Appendix.

Also:

$$\begin{aligned}
\text{V-comp. } OF_1 &= OF_1 \times \sin\alpha = 11 \times 0.866 = 9.53 \\
\text{V-comp. } OF_2 &= OF_2 \times \sin\beta = 18 \times 0.5 \quad = 9.00 \\
\text{V-comp. } OF_0 &= OF_0 \times \sin\theta = \text{the sum} \quad = 18.53
\end{aligned}$$

$$OF_0 = \sqrt{21.1^2 + 18.53^2} = \sqrt{445 + 343.5} = 28.1 \text{ lb.},$$

$$\tan\theta = \frac{18.53}{21.1} = 0.878 \quad \theta = 41^\circ \text{ approximately.}$$

When more than two vectors are to be added, the sum of all of the vertical and horizontal components are used. Thus:

$$\text{Resultant} = \sqrt{\left(\sum\text{H-comps.}\right)^2 + \left(\sum\text{V-comps.}\right)^2}.$$

Where

$$\sum\text{H-comps.} = F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \cos\alpha_3 \ldots$$

and

$$\sum\text{V-comps} = F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + F_3 \sin\alpha_3 \ldots$$

$$\tan\theta = \frac{\Sigma\text{V-comps.}}{\Sigma\text{H-comps.}}.$$

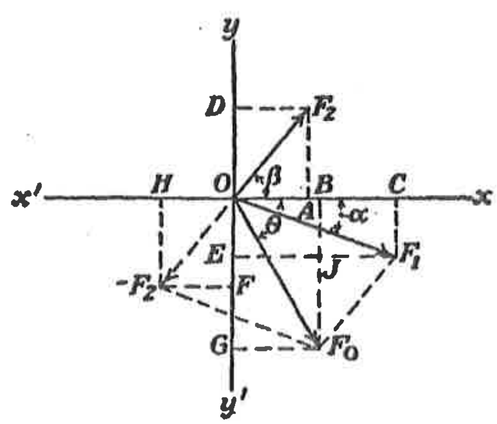


FIG. 25.—Vector difference—parallelogram method.

If the difference of two vectors is required, the process is essentially the same as the above, except that the difference instead of the sum of the components is taken.

Thus in Fig. 25 the difference of two vectors $(F_1 - F_2)$ is found by method 1 of Art. 12, Part I.

We see from the geometry of the figure that

$$F_1F_0 = -OF_2.$$

Therefore:

$$OA = -OH = BC.$$

Hence:

$$OB = OC - BC = OC - OA.$$

Thus:

$$\text{H-comp. } F_0 = \text{H-comp. of } F_1 - \text{H-comp. } F_2.$$

Also:

$$-F_2F_0 = OF_1.$$

Therefore:

$$-OD = HF_2 = JF_0 = EG.$$

Hence:

$$OG = OE + EG = OE - OD.$$

Thus:

$$\text{V-comp. } F_0 = \text{V-comp. of } F_1 - \text{V-comp. } F_2,$$

$$F_0 = \sqrt{(\text{H-comps. } F_0)^2 + (\text{V-comp. } F_0)^2} \text{ and}$$

$$\tan \Theta = \frac{\text{V-comp. } F_0}{\text{H-comp. } F_0} \text{ as before.}$$

As a numerical example, suppose that $F_1 = 12$ pounds and $\alpha = 20$ degrees. Also that $F_2 = 8$ pounds and $\beta = 50$ degrees. From a table of trigonometric functions we find that $\cos(-\alpha) = 0.94$, $\sin(-\alpha) = -0.34$, $\cos \beta = 0.64$, and $\sin \beta = 0.766$.

Therefore:

$$\begin{aligned} \text{H-comp. } F_1 &= 12 \times 0.94 &&= 11.30 \\ \text{H-comp. } F_2 &= 8 \times 0.64 &&= 5.12 \\ \hline \text{H-comp. } F_0 &= \text{the difference} &&= 6.18 \end{aligned}$$

Also:

$$\begin{aligned} \text{V-comp. } F_1 &= 12 \times (-0.34) &&= -4.08 \\ \text{V-comp. } F_2 &= 8 \times 0.766 &&= 6.13 \\ \hline \text{V-comp. } F_0 &= \text{the difference} &&= -10.21. \end{aligned}$$

$$F_0 = \sqrt{6.18^2 + (-10.21)^2} = \sqrt{38.19 + 104.3} = 11.94,$$

$$\tan \theta = \frac{-10.21}{6.18} = -1.66$$

$$\theta = -59°.$$

The above example illustrates the proper method o designating negative components. This is done by prefixing the proper sign before the trigonometric function. Since the vertical component of F_1 is below the horizontal axis, it must be given a negative sign, as -4.08. Also the vertical component of the resultant F_0 is below the line and is therefore designated as -10.21. Since the square of a negative number is always positive, F_0 will always come out positive. However, the tangent may be negative, as -1.66. When this occurs, it means that the angle is negative, *i.e.*, in the second or fourth quadrant[1] (see Fig. 24*a*); care must be exercised in keeping track of signs.

If two or more vectors are to be subtracted from one vector, it is not necessary, by this method, first to add

[1] See Appendix.

those to be subtracted. Thus, if F_1, F_2, and F_3 are all to be subtracted from F, we have

$$\begin{aligned}\text{H-comp. } F_0 &= \text{H-comp. } F - \text{H-comp. } F_1 \\ &\qquad - \text{H-comp. } F_2 - \text{H-comp. } F_3 \ldots \\ &= \text{H-comp. } F - \Sigma\text{H-comps. } F_1, F_2, \\ &\qquad F_3, \ldots\end{aligned}$$

$$\begin{aligned}\text{V-comp. } F_0 &= \text{V-comp. } F - \text{V-comp. } F_1 - \text{V-comp.} \\ &\qquad F_2 - \text{V-comp. } F_3 \ldots \\ &= \text{V-comp. } F - \Sigma\text{V-comps. } F_1, F_2, \\ &\qquad F_3, \ldots\end{aligned}$$

Then

$$F_0 = \sqrt{\left(\text{H-comp. } F - \sum\text{H-comps. } F_1, F_2, F_3\right)^2 + \left(\text{V-comp. } F - \sum\text{V-comps. } F_1, F_2, F_3\right)^2}$$

and

$$\tan\theta = \frac{\text{V-comp. } F - \Sigma\text{V-comps. } F_1, F_2, F_3}{\text{H-comp. } F - \Sigma\text{H-comps. } F_1, F_2, F_3}.$$

CHAPTER IV

THE VECTOR DIAGRAM

21. Sine Waves.

If a single coil or loop of wire is rotated at constant velocity in a uniform magnetic field, there will be generated in the coil an alternating e.m.f. Thus, if in Fig. 26*a*, the coil *ab* rotates about an axis perpendicular to the paper in a counterclockwise direction at constant angular velocity ω (omega) in a uniform

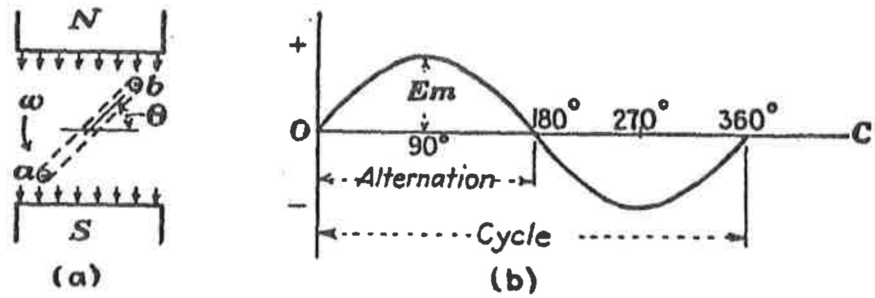


FIG. 26.—Method of generating a sine wave e.m.f.

magnetic field $N - S$, there will be generated in the coil a sine-wave e.m.f. The magnitude of this e.m.f. at any instant will be proportional to the sine of the angle θ. The e.m.f. when θ = zero, that is, the coil *ab* in a horizontal position, is zero. When $\theta = 90$ degrees and the coil is in a vertical position, the e.m.f. is a positive maximum. When the coil is again horizontal, at $\theta = 180$ degrees, the e.m.f. is again zero. When the coil is again vertical, at $\theta = 270$ degrees, the e.m.f. is a negative maximum. Finally, when the coil reaches its starting point, and $\theta = 360$

degrees, the e.m.f. is again zero. If, now, we plot the e.m.f. for successive positions of the coil, a smooth curve called a sine wave will result, as in Fig. 26*b*. In this curve the maximum value that the curve attains is called the maximum value of the e.m.f. and is designated as *Em*. One loop, from zero to the next zero value, is called an alternation; and one complete set of values, a cycle. In a 60-cycle system 1 cycle occurs in 1/60 second.

The same curve might have been plotted from a table of sines. If we lay off a horizontal axis *OC* in degrees and plot the value of the sine corresponding to these degrees vertically, a sine wave will result. Thus:

θ	Sin θ	θ	Sin θ	θ	Sin θ
0	0.000	120	0.886	240	−0.866
30	0.500	150	0.500	270	−1.000
60	0.866	180	0.000	300	−0.866
90	1.000	210	−0.500	330	−0.500
..	.....	...	....	360	0.000

The same curve may be produced graphically. To do this draw a circle of radius *A* and divide the circumference into any number of equal parts, say 12 (see Fig. 27). Next draw a horizontal line *OA* which, if extended, would pass through the center of the circle. Divide this line into the same number of equal parts. Erect perpendiculars at these points. Project the divisions on the circle upon the corresponding perpendiculars by horizontal lines. Finally draw a smooth curve through the intersection of these projections with the perpendiculars. A sine wave will result.

For various reasons, commercial alternators may not generate pure sine waves. However, they are usually sufficiently close to warrant being treated as such. For this reason considerable space has just been devoted to the various means of obtaining sine waves. The graphical method of Fig. 27 leads us directly into the next subject, *i.e.*, the representation of alternating voltages and currents by vectors.

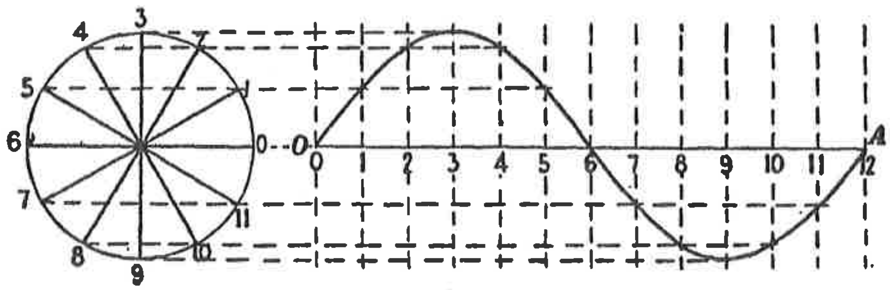


Fig. 27.—Graphical construction of a sine wave.

22. Vector Representation of Alternating Voltages and Currents.

If the 12 radii of Fig. 27 are replaced by a single radius of the same length, and this radius is caused to revolve at constant angular velocity ω, its projections upon a vertical line at any instant will give the value of the sine wave at that instant whose maximum value is equal to the length of the radius. If this radius, to some arbitrary scale, represents the maximum value of an alternating e.m.f., then it will generate the true sine wave of this e.m.f. For example, suppose that the line E in Fig. 28 is drawn to scale to represent the maximum value of an e.m.f. The instantaneous value of that e.m.f. may be found by projecting the line E upon equally spaced ordinates corresponding to successive positions of E, as was done in Fig. 27. If the voltage represented by E in Fig. 28 is a 60-cycle e.m.f. of 100 volts, the vector E must make 60 complete revolutions per second. At the instant shown,

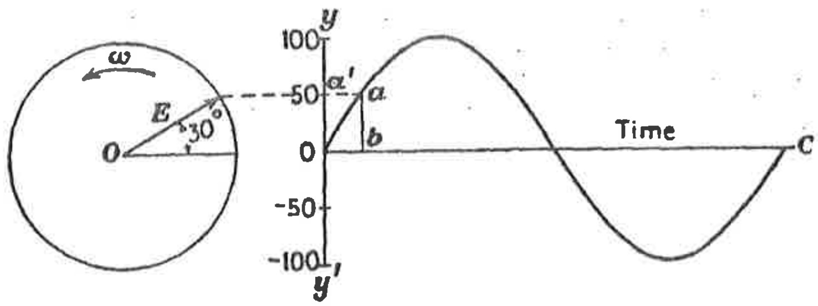


Fig. 28.—Instantaneous value of a voltage from a rotating vector.

the value of the e.m.f. is $ab = a'o = 100 \sin 30$ degrees $= 50$ volts. Since the vector E rotates at constant velocity, it traverses a fixed number of degrees per second. Therefore the axis OC is also

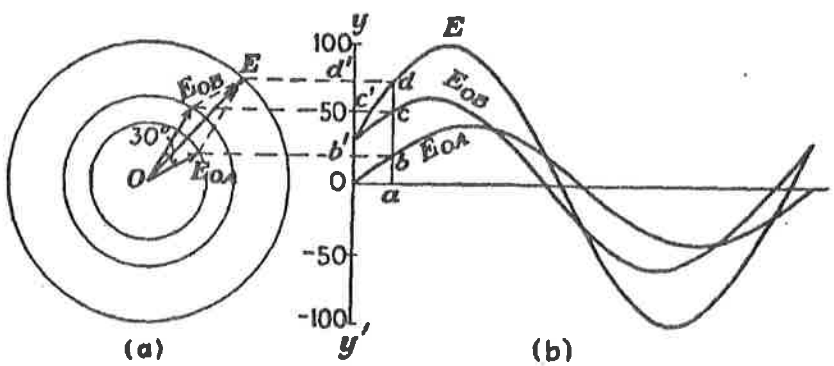


Fig. 29.—Relation of vector addition of voltages to the scalar addition of their sine-wave ordinates.

proportional to time; *i.e.*, time progresses from left to right along the axis OC. The e.m.f. reaches its maximum value at 90 degrees, this point corresponding at a frequency of 60 cycles, to $\frac{1}{60} \times \frac{1}{4} = \frac{1}{240}$ second after the start. In like manner, it passes through zero in the negative direction at 180 degrees, or $\frac{1}{60} \times \frac{1}{2} = \frac{1}{120}$ second after the start.

In Fig. 29*b*, we have plotted two sine-wave e.m.fs., E_{OA} and E_{OB}, differing in phase by 30 degrees. Remembering that time progresses from left to right, we see that E_{OB} leads E_{OA}, since it reaches its positive maximum before E_{OA} reaches its positive maximum. If now we add the ordinate of E_{OB} to the ordinate of E_{OA} point for point along the E_{OA} curve, we shall obtain their sum. This new curve E is also a sine-wave curve but it will not be in phase with either E_{OA} or E_{OB}. Alternating voltages and currents might always be added in this manner but the labor involved in drawing the individual sine waves and then adding their ordinates would be prohibitive.

Fortunately, we have a simpler method. In Fig. 29*a* the voltages E_{OA} and E_{OB} are represented by vectors and their sum is determined by completing the parallelogram. It will be noticed that at the instant shown, corresponding to the point *a* on diagram *b*, the sum of the ordinates equals the sum of the projections of the vectors. In other words,

$$ab + ac = ad = ob' + oc' = od'.$$

Thus the vector E truly represents the sum of E_{OA} and E_{OB} at the instant shown. If now the system of vectors in Fig. 29*a* rotates in unison, their projections will generate the identical three waves of Fig. 29*b*, and the value of the various e.m.fs. may be determined at any particular instant by stopping the rotation at that instant and projecting the ends of the vectors upon the line YY'.

The process of subtracting two sine-wave quantities is illustrated in Fig. 30*b*. Here E_{OB} leads E_{OA} by 60 degrees and

$$E_{OA} - E_{OB} = E.$$

The curve E is obtained by subtracting the ordinate of E_{OB} from the corresponding ordinate of E_{OA}. It should be borne in mind that the subtraction of a negative quantity is addition of the same positive quantity. The curve $-E_{OB}$ is also shown. Figure

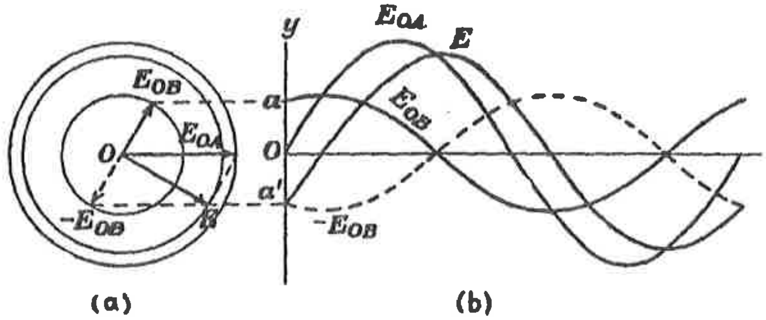


FIG. 30.—Relation of vector subtraction of voltages to scalar subtraction of their sine-wave ordinates.

30*a* shows the same process vectorially. At the instant illustrated,

$$E_{OA} = O,$$

therefore:

$$E = E_{OA} - E_{OB} = -E_{OB}.$$

Projecting the vectors upon the YY' axis, we find this to be true, or, in other words,

$$oa' = -oa.$$

23. Substitution of Effective Value for Maximum Value.

As stated in Chap. II, Art. 11, it is not essential that the vectors represent maximum values only. They may represent effective values. This is usually done, since this value is the one most generally employed in electrical problems. It is also the value recorded by most instruments. It can be proved by higher mathematics that the effective value equals the maximum value divided by the square root of two.[1] Thus

$$E(\text{eff.}) = \frac{Em}{\sqrt{2}}.$$

Since there is a fixed and definite ratio $\left(\frac{1}{\sqrt{2}}\right)$ between the effective and maximum values, the vectors of Fig. 29*a* would change only in length if they were to represent effective instead of maximum values. Their phase relations would not be affected. Furthermore, since this ratio depends upon the maximum value only, it is not essential that we consider the set of vectors in Fig. 29*a* as rotating. We may consider them as stationary for the purpose of problem analysis. In general, what can be done with instantaneous values (sine waves) can be done with vector values in the same sense, in so far as addition and subtraction are concerned. It is only essential that the various quantities be of the same frequency.

24. Notations.

The notations, rules, and conventions used in connection with our symbols of Part I may be used without change for our vectors. This includes the subscript notation used throughout Part I, the counterclockwise phase rotation, the arrowhead convention of Art. 8, and the rules of Art. 17. Also multiplication

[1] Blalock, G. C., "Principles of Electrical Engineering," Chap. XIV, Art. 152.

is accomplished in an identical manner. In addition to what was said in Part I concerning subscripts, the following illustration of the effect of the interchange of subscripts will help to explain the subscript theory. Since E_{AO} designates a vector the positive value of which acts from O to A, and E_{AO} a vector whose positive value acts from A to O, it follows that

$$E_{OA} = -E_{AO}.$$

If we have two vectors, E_{AB} and E_{CD}, and wish to designate their sum, we can do so in two ways as follows:

$$\dot{E}_{AB} + \dot{E}_{CD} = \dot{E}_{AD} \text{ and } \dot{E}_{CD} - \dot{E}_{BA} = \dot{E}_{AD}.$$

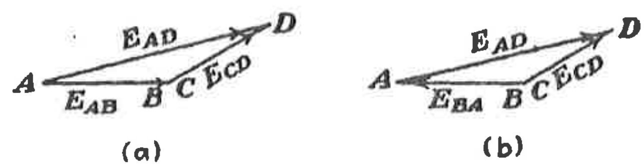


Fig. 31.—The effect of interchanging subscripts in an addition.

The dots over the symbols designate vectors in distinction to scalars. The above expression is a vector, not a scalar, sum.

The first equation is illustrated in Fig. 31*a* using the triangular method. Thus, going from A to B to C to D we obtain the sum of $E_{AB} + E_{CD} = E_{AD}$. In Fig. 31*b* are shown the vectors E_{BA} and E_{CD}. By the triangular method, going from A to B to C to D, we obtain the difference

$$-\dot{E}_{BA} + \dot{E}_{CD} = \dot{E}_{AD},$$

the same vector as in Fig. 31*a*. This may also be shown as a subtraction, thus:

$$\dot{E}_{AB} - \dot{E}_{CD} = \dot{E}_{DB},$$

and

$$\dot{E}_{DC} - \dot{E}_{AB} = \dot{E}_{DB}.$$

These equations are illustrated in Fig. 32*a* and *b*, respectively. The triangular method is used.

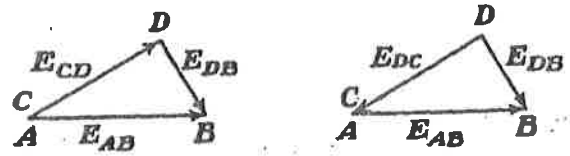


Fig. 32.—The effect of the interchange of subscripts in a subtraction

25. Examples.

The following examples will illustrate the principles discussed in Part II. The relative accuracy of the graphical and analytical solutions should be carefully noted. It will be seen that the graphical solution is sufficiently accurate for practical purposes.

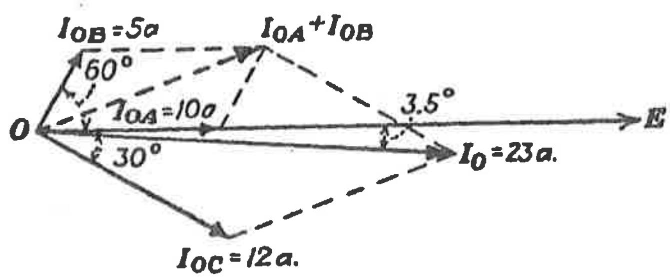


Fig. 33.—Vector diagram for example 1.

1. In a certain circuit there are three currents $I_{OA} = 10$ amperes in phase with the voltage, $I_{OB} = 5$ amperes leading the voltage by 60 degrees, and $I_{OC} = 12$ amperes lagging behind the voltage by 30 degrees. Find their sum vectorially and check analytically.

The vector diagram of this problem is shown in Fig. 33. The scale is 1 inch $= 16$ amperes and the value of the resultant current is $I_0 = 23$ amperes lagging behind the voltage by $\theta = 3.5$ degrees.

The analytical solution is obtained as follows, using Eqs. (6) and (7) of Art. 20:

H-comps.

$$I_{OA} \cos O = 10 \times 1.0 = 10.0$$
$$I_{OB} \cos 60 = 5 \times 0.5 = 2.5$$
$$I_{OC} \cos (-30) = 12 \times 0.866 = 10.4$$
$$I_o \cos \theta = \sum \text{H-comps.} = 22.9$$

V-comps.

$$I_{OA} \sin O = 10 \times 0.0 = 0.00$$
$$I_{OB} \sin 60 = 5 \times 0.866 = 4.33$$
$$I_{OC} \sin (-30) = 12 \times -0.5 = -6.00$$
$$I_o \sin \theta = \sum \text{V-comps.} = -1.67$$

$$I_0 = \sqrt{\left(\sum \text{H-comps}\right)^2 + \left(\sum \text{V-comps.}\right)^2} = \sqrt{524 + 2.79} = 22.95 \text{ amp.}$$

$$\tan \theta = \frac{\Sigma \text{V-comps.}}{\Sigma \text{H-comps.}} = -0.0729 \quad \theta = 4.1°.$$

2. With the same vectors as in Example 1, obtain the difference $\dot{I}_{OC} - \dot{I}_{OA} - \dot{I}_{OB}$ vectorially and check analytically.

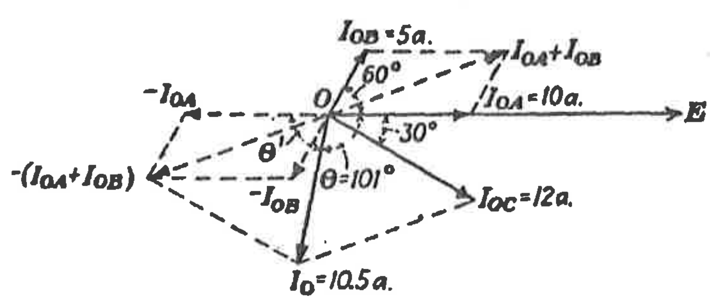


Fig. 34.—Vector diagram for example 2.

The vector diagram is shown in Fig. 34. A smaller scale is used in Fig. 34. Note that for the purpose of showing the signs of the components, it is best to reverse the vectors to be subtracted before adding. This is not essential, as is seen in the analytical solution where the original angles are used and the negative sign placed before the whole component to be subtracted.

H-comps.

$$\begin{aligned}
-I_{OA} \cos 0 &= -10 \times 1.0 &&= -10.0 \\
-I_{OB} \cos 60 &= -5 \times 0.5 &&= -2.5 \\
I_{OC} \cos(-30) &= 12 \times 0.866 &&= \underline{\ 10.4\ } \\
I_o \cos \theta' &= \textstyle\sum \text{H-comps.} &&= -2.1
\end{aligned}$$

V-comps.

$$\begin{aligned}
-I_{OA} \sin 0 &= -10 \times 0.0 &&= 0.00 \\
-I_{OB} \sin 60 &= -5 \times 0.866 &&= -4.33 \\
I_{OC} \sin(-30) &= 12 \times -0.5 &&= \underline{-6.00} \\
I_o \sin \theta' &= \textstyle\sum \text{V-comps.} &&= -10.33
\end{aligned}$$

$$I_o = \sqrt{\left(\textstyle\sum \text{H-comps.}\right)^2 + \left(\textstyle\sum \text{V-comps.}\right)^2} = \sqrt{4.42 + 106.7} = 10.52 \text{ amp.},$$

$$\tan \theta' = \frac{\Sigma \text{V-comps.}}{\Sigma \text{H-comps.}} = \frac{-10.33}{-2.1} = 4.92 \quad \theta' = 78.5°,$$

$$\theta = 180° - 78.5° = 101.5°.$$

Note that in the above solution, the angle θ′ is first found from which θ may be later determined.[1] Also that determining large angles geographically is more accurate than small angles.

3. Suppose a series circuit to consist of a pure resistance (non-inductive), a pure inductive reactance

[1] See Appendix.

(no resistance), and a pure capacity reactance. The voltage across the inductive reactance is $E_{OA} = 100$ volts, across the resistance is $E_{OB} = 80$ volts, and across the capacity is $E_{OC} = 150$ volts. Find their sum vectorially and check analytically.

Applying Rules *a*, *b*, and *c*, Art. 17, Part I, we draw E_{OA} 90 degrees ahead of the current, E_{OB} in phase with

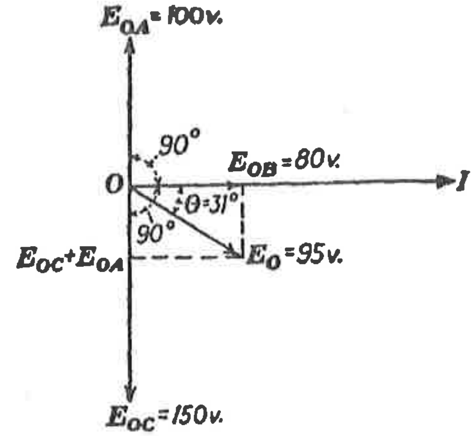


FIG. 35.—Vector diagram for example 3.

current, and E_{OC} 90 degrees behind the current. This is done in Fig. 35. The scale is 1 inch = 200 volts.

H-comps.

$$\begin{aligned} E_{OA} \cos 90 &= 100 \times 0.0 &&= 0.0 \\ E_{OB} \cos 0 &= 80 \times 1.0 &&= 80.0 \\ E_{OC} \cos(-90) &= 150 \times 0.0 &&= \underline{0.0} \\ E_o \cos \theta &= \textstyle\sum \text{H-comps.} &&= 80.0 \end{aligned}$$

V-comps.

$$\begin{aligned} E_{OA} \sin 90 &= 100 \times 1.0 &&= 100.0 \\ E_{OB} \sin 0 &= 80 \times 0.0 &&= 0.0 \\ E_{OC} \sin(-90) &= 150 \times -1.0 &&= -150.0 \\ E_o \sin \theta &= \textstyle\sum \text{V-comps.} &&= -50.0 \end{aligned}$$

$$E_O = \sqrt{\left(\sum \text{H-comps.}\right)^2 + \left(\sum \text{V-comps.}\right)^2} = \sqrt{6{,}400 + 2{,}500} = 94.3 \text{ volts},$$

$$\tan \theta = \frac{\Sigma\text{V-comps.}}{\Sigma\text{H-comps.}} = -\frac{50}{80} = -0.625, \ \theta = -32°.$$

If, in the above example, E_{OA} had been equal to E_{OC}, the sum of the three voltages would have been equal to E_{OB} and hence in phase with the current. This condition is termed resonance. Resonance is obtained in a radio set when properly tuned. Besides voltage or series resonance, we have also current resonance in some parallel circuits. In current resonance the vertical components cancel, leaving only the horizontal components in phase with the voltage. It sometimes happens, when resonance occurs, that dangerously high voltages or currents may appear in parts of the circuit even though the source of voltage or current is nominal.

4. A single-phase, 10-horse power, 86.3 per cent efficient induction motor is operating at 80 per cent power factor from a 110-volt, 60 cycle line. Also connected to this line are 10,000 watts of incandescent lamps. What are the line current and power factor?

$$\text{Power input to motor} = \frac{\text{output}}{\text{eff.}} \times 746 = \frac{10}{0.863} \times 746 = 8{,}650 \text{ watts} \quad (8)$$

where

746 = number of watts per horse power.

Since

$$P_M = E_M I_M \cos \theta_M,$$

the current drawn by the motor will be

$$I = \frac{P}{E_M \cos \Theta_M} = \frac{8{,}650}{110 \times 0.8} = 98.3 \text{ amp.}$$

Current drawn by lamps (unity power factor) =

$$\frac{P}{E} = \frac{10{,}000}{110} = 91 \text{ amp.}$$

$$\cos \Theta_M = 0.8.$$
$$\Theta_M = 37°.$$

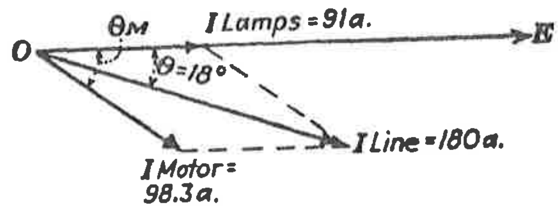


Fig. 36.—Vector diagram for example 4.

The vector diagram is shown in Fig. 36. Scale 1 inch = 160 amp.

H-comps.

$$I_{lamps} \cos 0 = 91 \times 1.0 = 91.00$$
$$I_{motor} \cos 37 = 98.3 \times 0.8 = \underline{78.64}$$
$$I_{line} \cos \Theta = \sum \text{H-comps.} = 169.64$$

V-comps.

$$I_{lamps} \sin 0 = 91 \times 0 = 00.0$$
$$I_{motor} \sin 37 = 98.3 \times 0.6 = \underline{59.0}$$
$$I_{line} \sin \Theta = \sum \text{V-comps.} = 59.0$$

$$I_{line} = \sqrt{\left(\sum \text{H-comps.}\right)^2 + \left(\sum \text{V-comps.}\right)^2} =$$
$$\sqrt{28{,}764 + 3{,}481} = 179.5 \text{ amp.,}$$

$$\tan \theta = \frac{\Sigma\text{V-comps.}}{\Sigma\text{H-comps.}} = \frac{59}{169.6} = 0.348.$$

$$\cos \theta = 0.94.$$

$$\theta = 19°.$$

Since

$$I_{\text{line}} \cos \theta = \Sigma\text{H-comps.},$$

we have that

$$\cos \theta = \frac{\Sigma\text{H-comps.}}{I_{\text{line}}} = \frac{169.64}{179.5} = 0.94.$$

This is a shorter method where only the power factor is required.

26. Problems

1. A certain load consists of three currents I_{OA}, I_{OB}, and I_{OC}. $I_{OA} = 20$ amperes and lags behind the voltage by 30 degrees, $I_{OB} = 15$ amperes and lags behind the voltage by 45 degrees, and $I_{OC} = 10$ amperes and lags behind the voltage by 60 degrees. Find the sum vectorially and check analytically.

2. Three voltages $E_{OA} = 100$, $E_{OB} = 50$, and $E_{OC} = 75$ are each 120 degrees out of phase with each other. Find their sum vectorially and check analytically.

3. I_{OA}, a current of 100 amperes, is 90 degrees ahead of the voltage, another current $I_{OB} = 50$ amperes is in phase with the voltage, and a third current $I_{OC} = 90$ amperes is 90 degrees behind the voltage. Draw the vector diagram. What is the sum of these currents and its phase position? Has resonance been obtained?

4. A voltage of 100 volts is 45 degrees out of phase with another voltage of 80 volts. Find the difference between the former and the latter vectorially and check analytically.

5. Three equal voltages are 120 degrees out of phase with each other. Determine their sum vectorially and check analytically. NOTE.—This is the condition existing in a symmetrical three-phase system.

6. Draw a sine wave graphically whose maximum value equals 2 inches.

7. Repeat Prob. 6 using a table of sines.

8. By means of rotating vectors, draw two sine-wave currents of 10 and 6 amperes maximum value differing in phase by 45 degrees. The 10-ampere current leads.

9. Obtain the sum of the two currents in Prob. 8 by adding ordinates and check with the vector sum of the rotating vectors.

10. Draw three equal sine-wave currents differing in phase by 120 degrees and obtain their sum by adding ordinates. Note.—This is the condition existing in a balanced three-phase system.

PART III

VECTOR SOLUTIONS OF POLYPHASE PROBLEMS

CHAPTER V

POLYPHASE SYSTEMS

We have dealt thus far with the theory of the vector and its manipulation in the representation of alternating voltages and currents. The examples and problems used have been largely for single-phase circuits. This was done because such problems are simpler and illustrate equally well the application of vector theory. The real field for vector analysis, however, is in polyphase circuits. Here the more or less intricate relation of a number of quantities makes it imperative that we have a means of visualizing as well as solving polyphase problems. The relatively large number of proofs, problems, and solutions illustrated in this part of the text, although typical, cannot but scratch the surface of those that might arise. For this reason, the reader should pay particular attention to the method of attack and the technique of analysis so that he can apply them to the solution of his own particular problems with assurance and confidence.

27. Classification of Polyphase Systems.

Before undertaking the solution of polyphase problems, it will be well to review a few of the fundamental principles of the polyphase circuit. A polyphase circuit is one energized by more than one e.m.f. These e.m.fs. are usually symmetrical, that is, equal in magnitude and differing in phase by equal amounts. Although in practice they may differ slightly either in magnitude or in phase position, they are almost always sufficiently close to warrant their treatment as such. Common polyphase systems are:

I. Two phase.
   a. Four wire.
   b. Three wire.
   c. Five wire.

II. Three phase.
   a. Three wire.
   b. Four wire.

Since the three-phase, three-wire and three-phase, four-wire systems are most common, they alone will be considered.

28. Generation of Three-phase Voltages.

In Chap. I, Art. 4, Fig. 3, we learned that there are two ways in which the coils may be placed on an alternator to produce symmetrical three-phase voltages 120 degrees apart.

If we plot the instantaneous values of the three voltages generated in the coils of Fig. 3 for either the *Y* or the delta connection, we shall obtain the three sine waves of Fig. 37*b*. In this figure the voltage generated in coil *OA* is zero and about to increase in the positive direction when the time $t = 0$. Obviously, the voltage in the coil *OB* will not pass through a similar point until 120 electrical time degrees later. In like manner, the voltage in coil *OC* will not pass through a similar point until 240 electrical time degrees after E_{OA}. However, the point of particular significance to be derived from Fig. 37*b* is that at any instant the algebraic sum of the voltages is zero. This may be proved by adding ordinates at any point. Thus, when E_{OA} is zero, E_{OB} and E_{OC} are 86.6 per cent of their maximum values and opposite in sign. Therefore their algebraic sum is zero. When E_{OA} is a maximum, E_{OB} and E_{OC} are equal to half their maximum value and both opposite in sign to E_{OA}. Consequently, the algebraic sum is again zero. This relation will hold for any other point that might be selected. It may also be illustrated as in Fig. 37*c*. Here the vector sum of E_{OC} and E_{OB} is found by the parallelogram method to be equal and opposite to E_{OA}. Therefore the vector sum of E_{OC}, E_{OB}, and E_{OA} is zero.

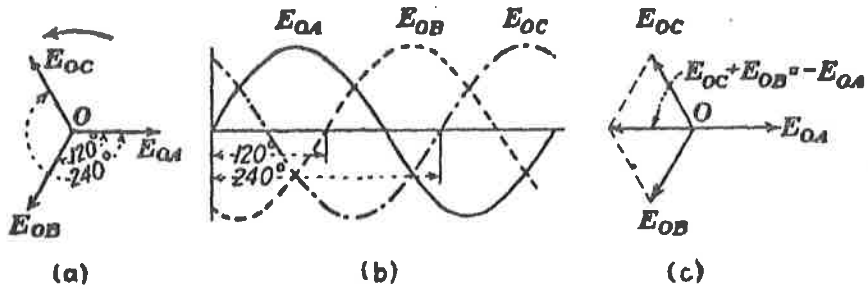


FIG. 37.—Three-phase voltage waves.

29. The Y-Connection.

As stated in Chap. I, Art. 4, the Y-connection is obtained by joining the three corresponding ends of the coils together at a common point O. If three line wires are attached to the other ends of the three coils, a three-phase, three-wire system results. If a fourth wire, called the neutral, is attached at the point O and also carried along with the other three, a three-phase, four-wire system results. The voltages actually generated in the coils themselves are called the coil or Y voltages to distinguish them from the line voltages. The relation of these two sets of voltages is shown in Fig. 38*b*. The vectors E_{OA}, E_{OB}, and E_{OC} are the coil voltages of the coils OA, OB, and OC of Fig. 38*a*. The line voltage E_{ab} (the voltage from line a to line b) is equal to the voltage from A to B, but to go from A to B we go against voltage E_{OA} and with voltage E_{OB}. Going against voltage E_{OA} is the same as going with E_{OA} reversed. In other words, we derive the line voltage E_{ab} by obtaining the vector difference $E_{OB} - E_{OA}$. The other two line voltages are found in a similar manner. The result is three new voltages equal in magnitude and 120 degrees apart but out of phase with the phase voltages by 30 degrees. The actual magnitude may be obtained by the method of Chap. II, Art. 20. For convenience, consider Fig. 38*c*. This figure shows the parallelo-

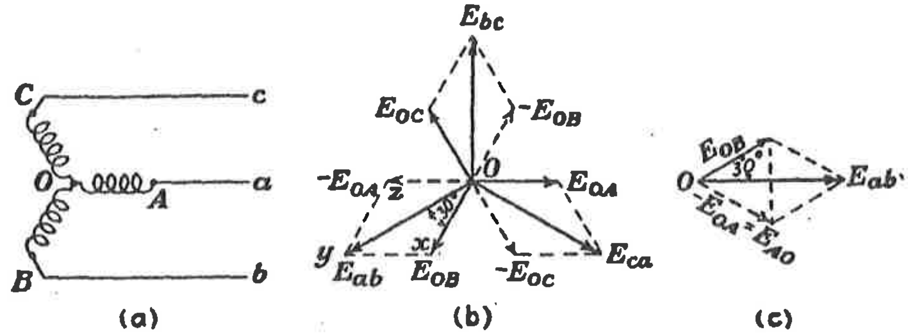


FIG. 38.—Coil and line voltage relations for a Y-connection.

gram *oxyz* of Fig. 38*b* by itself. The magnitude of E_{ab} is found as follows:

H-comps.

$$E_{OB} \cos 30 = 0.866 E_{OB}$$
$$E_{AO} \cos(-30) = 0.866 E_{AO}$$

$$\text{H-comps. } E_{ab} = 0.866(E_{OB} + E_{AO})$$

V-comps.

$$E_{OB} \sin 30 = 0.5 E_{OB}$$
$$E_{AO} \sin(-30) = -0.5 E_{AO}$$

$$\text{V-comps. } E_{ab} = 0$$
$$E_{ab} = 0.866 \times (2E_{OB}), \quad \text{since } E_{AO} \text{ numerically equals } E_{OB}.$$

Therefore

$$E_{ab} = 1.73E_{OB},$$

since there is no resultant Y-comp. Thus, the line voltage equals 1.73 times the coil voltage in a Y connection, but 1.73 is also equal to the $\sqrt{3}$.

Rule 1.—*In a balanced, symmetrical, three-phase, Y-connected system, the line voltages are all equal and* 120 *degrees apart. Each is* 30 *degrees out of phase with one of its respective coil voltages. These line voltages are each* $\sqrt{3}$, *or* 1.73 *times the coil voltage.*

If a current flows in line *Aa*, the same current must flow in the coil *OA*, since it is in series with the line. In like manner, the other line currents are equal to their respective coil currents. Moreover, the vector sum of these three line currents is zero, provided there is no neutral wire. This may be shown by the method of Fig. 37*b*, substituting currents for voltages, or, more simply, by the fact that the three coils meet in a common point *O*. If the vector sum of the three currents were not zero, there would be a loss or a gain of current at *O*, as in a leaky pipe. An insulated joint of wires cannot lose or gain current. Therefore the vector sum of the currents is zero. This holds whether the currents are equal or unequal, namely, the system balanced or unbalanced.

Rule 2.—*In a Y-connected system, the line and coil currents are respectively equal and their vector sum is zero.*

To distinguish between line and coil voltages, or line and coil currents, we shall use throughout the remainder of this text the word line and coil as a subscript. Thus E_{coil} means the actual coil or phase voltages, while E_{line} means the voltage between lines. Similarly, I_{coil} will refer to the current in a coil, while I_{line} will refer to a current flowing out in the line. This notation is used in the following paragraph and thereafter.

The power delivered by each coil is

$P_{coil} = E_{coil} I_{coil} \cos \theta_{coil}$. (Eq. (5), Chap. II, Art. 15.)

For a balanced system the three coils will furnish

$$P = 3E_{coil} I_{coil} \cos \theta_{coil}.$$

Since

$$E_{coil} = \frac{E_{line}}{\sqrt{3}},$$

and

$$I_{line} = I_{coil},$$

$$P = \frac{3}{\sqrt{3}} E_{line} I_{line} \cos \theta_{coil}$$

$$= \sqrt{3}\, E_{line} I_{line} \cos \theta_{coil}, \qquad (9)$$

and

$$\cos \theta_{coil} = \text{power factor} = \frac{P}{\sqrt{3}\, E_{line} I_{line}}. \qquad (10)$$

Rule 3.—*In a balanced three-phase, Y-connected system, the system power factor is the cosine of the angle between the coil voltage and coil current, and the power delivered by the system is*

$$P = \sqrt{3}\, E_{line} I_{line} \cos \theta_{coil}.$$

If the system is appreciably unbalanced, the system power factor has no significance, and the power is the sum of the three-phase powers whatever they may be.

However, the vector sum of the currents must still be zero.

In a balanced three-phase, four-wire system, the same rules apply, since there is no current in the fourth or neutral wire. If this system is unbalanced, the neutral carries the unbalanced current such that the vector sum of the four currents is zero. In this case the power delivered is the sum of the three-phase powers and the system power factor has no significance.

30. The Delta Connection.

As stated in Chap. I, Art. 4, the delta connection is obtained by connecting the coils in rotation. If

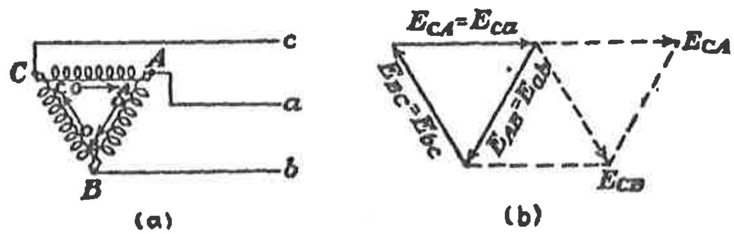


FIG. 39.—Coil and line voltage relations for a delta connection.

three line wires are now attached to the three junction points of the coils, a three-phase, delta-connected system results (see Fig. 39*a*).

In this figure the three coils of Fig. 38*a* are connected in rotation in the same order. The *O*s now are useless but are included to show the method of making connections. It might at first seem as if a current would flow around the coils. That this is not the case may be seen by the diagram in Fig. 39*b*. In this figure the three coil voltages E_{CA}, E_{AB}, and E_{BC} form the solid equilateral triangle. The dotted lines show the vector sum of E_{CA} and E_{AB}. This sum, E_{CB}, is equal and opposite to E_{BC}. Therefore the sum of the three voltages is zero and there can be no circulating current. It is obvious that in the delta connection the line and coil voltages are equal.

Rule 4.—In a balanced symmetrical three-phase, delta-connected system, the line voltages equal the coil voltages.

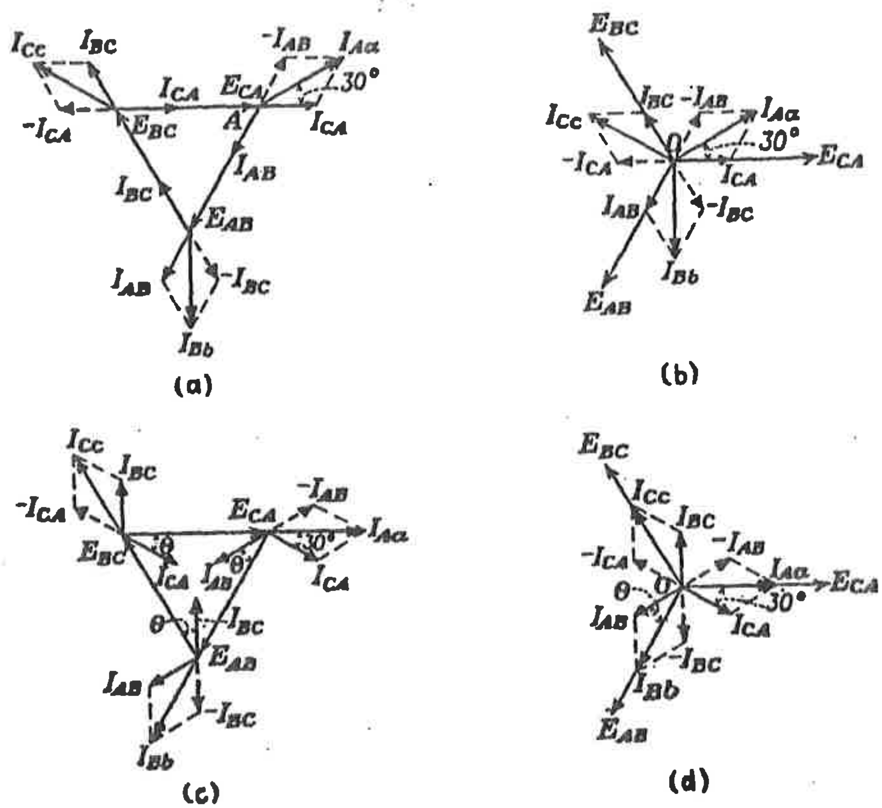


Fig. 40.—Coil and line current relations for a delta connection.

Since, as we have just seen, the sum of any two of the voltages of Fig. 39*b* is equal in magnitude to the third, it is possible to operate the system at reduced power without one of the coils. The result is what is called an open-delta connection. This system is often used where it is desired to save the expense of the third transformer.

In the Y connection the line voltage was equal to $\sqrt{3}$ times the coil voltage. We will now show that in a delta connection it is the line current that is equal to $\sqrt{3}$ times the coil current. In Fig. 40*a* we have the vector diagram for a balanced unity power factor delta-connected load. The middle triangle is the same voltage triangle of Fig. 39*b* to which has been added the three-coil currents I_{CA}, I_{AB}, and I_{BC}. These currents are in phase with their voltages, since the load is unity power factor. For convenience they are also shown moved out to the corners of the delta. Consider the corner *A* (Fig. 39*a*). At this point, coil current I_{CA} is flowing into A and coil current I_{AB} is flowing out of *A*. The current that flows out along the line *Aa* from *A* to *a* must, therefore, be the vector difference between I_{CA} and I_{AB}. This vector difference is shown as I_{Aa} (Fig. 40*a*). It will be noticed that this line current is out of phase with one of its coil currents by 30 degrees. Its magnitude, as in the case of the Y voltages, is $\sqrt{3}$ times the coil current. A similar argument holds at the other line wires. These line currents are also shown.

Figure 40*b* is vectorially the same as Fig. 40*a*. The voltages are moved parallel to themselves without other change to form an equivalent Y diagram. The line currents are then found by taking the difference of the proper coil currents as in Fig. 40*a*. It is often simpler to draw delta diagrams this way. It is less confusing to have the vectors all radiating from a common point *O*.

Figures 40*c* and *d* are a similar set of diagrams drawn for lagging currents. It will be noticed that the line currents are again 30 degrees out of phase with the coil currents and are therefore equal to the $\sqrt{3}$ times the coil currents.

Rule 5.—*In a balanced delta-connected system the line currents are* 30 *degrees out of phase with one of their respective coil currents and equal to* $\sqrt{3}$ *times the coil current.*

As in the case of the Y connection, the power delivered by each phase is

$$P_{coil} = E_{coil} I_{coil} \cos \theta_{coil}.$$

For a balanced system the three coils will deliver

$$P = 3E_{coil} I_{coil} \cos \theta_{coil}.$$

Since

$$E_{coil} = E_{line},$$

and

$$I_{coil} = \frac{I_{line}}{\sqrt{3}},$$

$$P = \frac{3}{\sqrt{3}} E_{line} I_{line} \cos \theta_{coil}$$
$$= \sqrt{3}\, E_{line} I_{line} \cos \theta_{coil},$$

and

$$\cos \theta_{coil} = \text{power factor} = \frac{P}{\sqrt{3}\, E_{line} I_{line}}.$$

These are the same equations as were derived from the Y connection.

Rule 6.—*In a balanced three-phase delta-connected system, the system power factor is the cosine of the angle between the coil voltage and the coil current, and the power delivered by the system is* $P = \sqrt{3}\, E_{line} I_{line} \cos \theta_{coil}$.

The same statements concerning unbalance hold for the delta system as did for the Y system.

CHAPTER VI

MEASUREMENT OF POWER IN A THREE-PHASE, THREE-WIRE SYSTEM

31. Two-wattmeter Method of Measuring Power in a Three-phase System.

To measure power in a three-phase, three-wire system whether Y or delta, two wattmeters are customarily used. These meters are connected as in Figs. 41*a* and *b*. Careful attention should be paid to this method of metering, since it is the basic method upon which all other schemes depend. The current coils of the two wattmeters are connected in two of the line wires and the potential coils from their respective line wires to the third line wire. This applies equally well to energy measurements by watthour meters. It is essential for an indicating wattmeter to give forward deflection or a watthour meter to give forward rotation that the current in the current coil and the voltage in the potential coil be in the same direction at the same instant, (see Fig. 41*e*). It makes no difference which direction. Consequently the first step in analyzing any wattmeter connection is to assume a direction of current flow. In Figs. 41*a* and *b* we have assumed current flow toward the load as indicated by the arrows parallel to the line wires and load coils. Bear in mind that we could just as well have assumed the current flowing in the opposite direction. However, having once assumed the current flowing toward the load, we have fixed the direction of current flow through the wattmeter in this case from left to right. In other words,

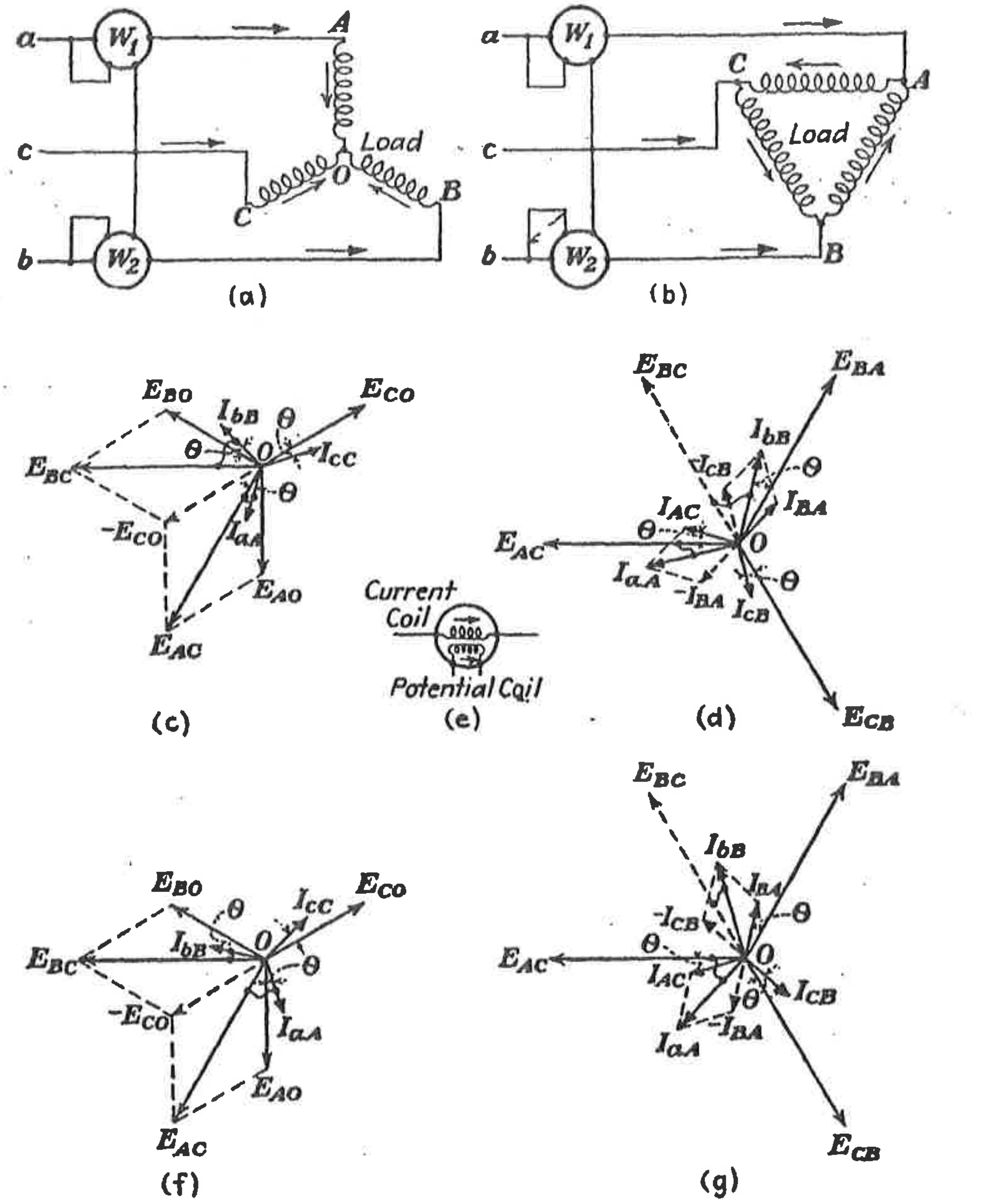


Fig. 41.—Two-wattmeter method of measuring power in a three-phase system.

the entering current and potential terminals are on the left of the meter. Consequently, for forward movement of the meter, we must apply to it a potential acting from left to right in the meter. For wattmeter

W_1 (Fig. 41*a*), this is the potential E_{AC}, and for wattmeter W_2 it is the potential E_{BC}.

The best way to show just which voltage and current, and their phase relation, are applied to each wattmeter is to draw the vector diagram for the circuit. This is done in Fig. 41*c*. Since we have assumed the current and the voltages acting toward the load, we draw E_{AO} vertically downward to correspond to the voltage acting from A to O in our circuit diagram. We draw E_{CO} to the right at 120 degrees and E_{BO} to the left 120 degrees for the same reason. In a Y-connected system the coil and line currents are equal and are drawn slightly lagging their respective coil voltages by the angle θ. As stated above, the voltage applied to W_1 is E_{AC}.

$$\dot{E}_{AC} = \dot{E}_{AO} - \dot{E}_{CO}$$

and is found by reversing E_{CO} and adding it to E_{AO}. Also, as previously stated, the voltage applied to W_2 is E_{BC}.

$$\dot{E}_{BC} = \dot{E}_{BO} - \dot{E}_{CO}$$

and is found by adding E_{CO} reversed to E_{BO}. The current in W_1 is I_{aA} and the voltage is E_{AC}. These two vectors are tied together by a strap to indicate that they are paired up in a given wattmeter. This indication will be employed throughout the remainder of this text. Also the current in W_2 is I_{bB} and the voltage is E_{BC}, and these are likewise tied together.

It will be noticed that neither current is in phase with its voltage. Even if the power-factor angle θ had been zero (unity power factor), they still would not be in phase. Only that part of the current that is in phase with the voltage is effective in actuating the wattmeter. This in-phase component is found by multiplying the current by the cosine of the angle between the current and the voltage. In the case of I_{aA} and E_{AC} this angle is equal to $(30 - \theta)$ degrees. We know from Rule 1, Art. 29, that E_{AC} is out of phase with E_{AO} by 30 degrees. I_{aA} is out of phase with E_{AO} by θ degrees. Therefore I_{aA} is out of phase with E_{AC} by $(30 - \theta)$ degrees. Similar reasoning will show that I_{bB} is $(30 + \theta)$ degrees out of phase with E_{BC}. It follows therefore that

$$W_1 = E_{AC}I_{aA} \cos (30 - \theta)$$

and

$$W_2 = E_{BC}I_{bB} \cos (30 + \theta).$$

If the system is balanced, then these equations reduce to

$$W_1 = E_{\text{line}}I_{\text{line}} \cos (30 - \theta). \qquad (11)$$

$$W_2 = E_{\text{line}}I_{\text{line}} \cos (30 + \theta). \qquad (12)$$

Equations 11 and 12 are the fundamental expressions for the wattmeters in the two-wattmeter method, or the two elements of a polyphase watthour meter for energy measurement. Unless the wattmeters read as indicated in Eqs. (11) and (12) they do not record the true power consumption. It can be shown by expanding the angles $(30 - \theta)$ and $(30 + \theta)$ by trigonometric formulas that

$$W_1 \pm W_2 = \sqrt{3}E_{\text{line}}I_{\text{line}} \cos \theta_{\text{coil}} \qquad (13)$$

the expression for the power in a three-phase circuit.[1]

[1] $W_1 = EI \cos (30 - \theta) = EI (\cos 30 \cos \theta + \sin 30 \sin \theta)$

and

$$W_2 = EI \cos (30 + \theta) = EI (\cos 30 \cos \theta - \sin 30 \sin \theta).$$

It may also be shown that the sum of the two-wattmeter readings will give the true power even if the system is unbalanced.[1]

Returning now to Fig. 41*d*, this diagram is the Y equivalent of the delta load of Fig. 41*b*. Currents are again assumed flowing toward the load. Thus the line voltages and therefore the coil voltages (since it is a delta-connected load) are acting from *A* to *C*, *C* to *B*, and *B* to *A*, respectively. Therefore we draw E_{AC} horizontal and to the left to correspond to our load coil *AC*. In the same way we draw E_{CB} 120 degrees down toward the right and E_{BA} 120 degrees up toward the right. The coil currents are then drawn lagging their respective coil voltages by the angle Θ. The line currents are obtained by subtracting the proper coil currents, as was done in Fig. 40. As before, W_1 has associated with it the line current I_{aA}, the voltage E_{AC}, and the angle $(30 - \Theta)$. W_2 has associated with

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Therefore:

$$\begin{aligned} W_1 + W_2 &= 2EI \cos 30 \cos \Theta \\ &= 2EI \frac{\sqrt{3}}{2} \cos \Theta \\ &= \sqrt{3}\, EI \cos \Theta \text{ (see also Appendix).} \end{aligned}$$

[1] Using instantaneous values, the total power is

$$p = e_{AO}i_{aA} + e_{BO}i_{bB} + e_{CO}i_{cC};$$
$$i_{aA} + i_{bB} + i_{cC} = 0 \text{ (Rule 2, Art. 29);}$$

Therefore:

$$i_{cC} = -(i_{aA} + i_{bB}).$$

Substituting:

$$p = e_{AO}i_{aA} + e_{BO}i_{bB} - e_{CO}i_{aA} - e_{CO}i_{bB} = (e_{AO} - e_{CO})i_{aA} + (e_{BO} - e_{CO})i_{bB},$$

$$W_1 = (e_{AO} - e_{CO})i_{aA} \qquad W_2 = (e_{BO} - e_{CO})i_{bB};$$

Therefore:

$$W_1 + W_2 = p.$$

it the current I_{bB}, and in order to produce forward movement it must have the voltage E_{BC}. E_{BC} is E_{CB} reversed. When reversed, this voltage makes an angle of $(30 + \theta)$ with I_{bB}. Therefore,

$$W_1 = E_{AC} I_{aA} \cos (30 - \theta).$$
$$W_2 = E_{BC} I_{bB} \cos (30 + \theta).$$

If the system is balanced, these expressions reduce to Eqs. (11) and (12) and therefore Eq. (13).

It should not be inferred that the angle $(30 - \theta)$ is always associated with W_1 and $(30 + \theta)$ with W_2. If the phase rotation is reversed or the currents are leading instead of lagging, they will be found to be interchanged. The important point is that both are present with one wattmeter or the other. The case of leading currents is illustrated in Figs. 41*f* and *g*. Inspection of these diagrams will readily show that

$$W_1 = E_{AC} I_{aA} \cos (30 + \theta).$$
$$W_2 = E_{BC} I_{bB} \cos (30 - \theta).$$

In this case the angle terms have been interchanged between wattmeters. However, if expanded and added as above, these expressions likewise reduce to Eq. 13.

32. Effect of Power Factor upon the Wattmeter Readings.

The plus or minus sign of Eq. 13 will now be explained. If the power factor is unity $(\theta = 0)$, then $W_1 = W_2$ (from Eqs. (11) and (12)). If the power factor is between unity and 50 per cent, W_1 will read more than W_2, but both will be positive. If the power factor is 50 per cent, $(\theta = 60)$,

$$P = \sqrt{3}EI\tfrac{1}{2} \text{ (from Eq. (13)},$$

$$W_1 = EI\frac{\sqrt{3}}{2} \text{ (from Eq. (11)},$$

and

$$W_2 = 0 \text{ (from Eq. (12))},$$

since

$$\cos(-30) = \frac{\sqrt{3}}{2}$$

and

$$\cos 90 \text{ degrees} = 0.$$

If the power factor is less than 50 per cent, θ is greater than 60 degrees and $(30 + \theta)$ becomes greater than 90 degrees and therefore $\cos(30 + \theta)$ is negative. Hence W_2 will read backward. Its potential leads must then be interchanged and its reading must be subtracted from W_1. If it is a watthour-meter element, it will tend to rotate backward, and the net forward torque of both elements will be the difference, instead of the sum, of the individual torques.

However, taking both elements of the watthour meter together or both wattmeters together, the net result is always positive for energy or power measurement. If the watthour meter as a whole rotates backward or the algebraic sum of the two wattmeters is negative at any power factor from unity to zero, they were not properly connected to measure energy or power respectively.

33. Wattmeter Readings at Power Factors of 50 Per Cent and Less.

The importance of Eqs. (11) and (12) cannot be too highly emphasized. Consequently it will be well to draw the vector diagrams for 50 per cent power factor and for a power factor less than 50 per cent. Using the same circuits and assumptions as in Fig. 41, we draw the same voltage vectors. The currents, however, lag by 60 degrees in the first case (Figs. 42*a* and *b* and lag 75 degrees in the second case (diagrams *c* and *d*).

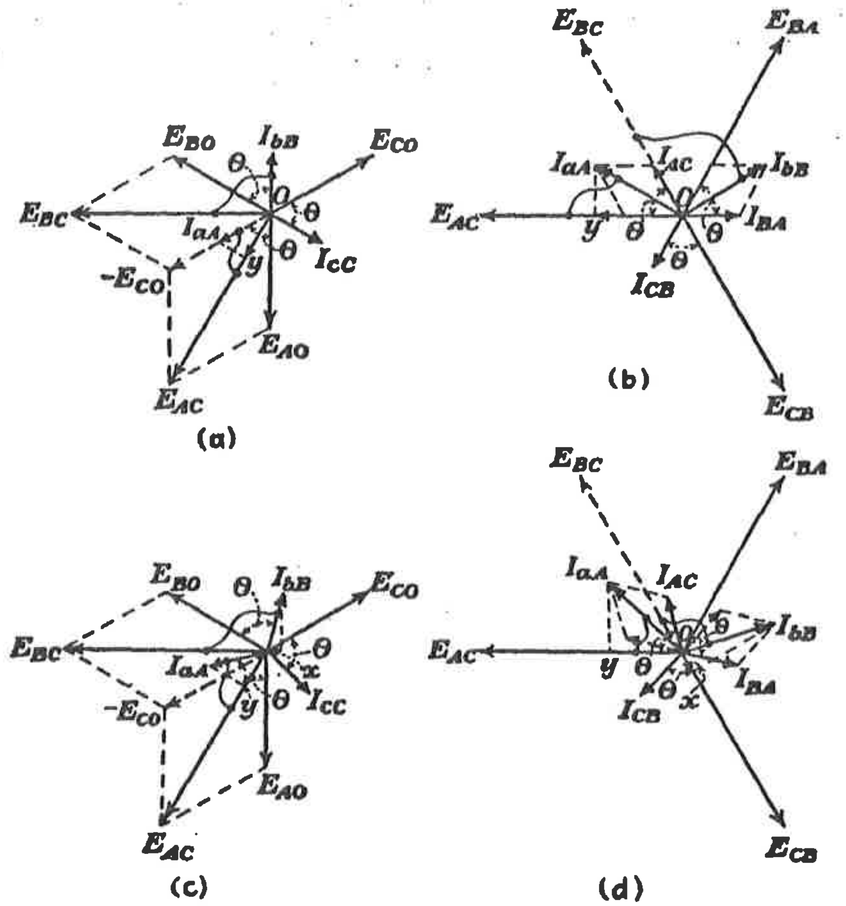

FIG. 42.—Two-wattmeter method—power factor 50 per cent and less than 50 per cent.

In these figures the same currents and voltages are associated with the same wattmeters as in Fig. 41. In Fig. 42*a*, I_{aA} lags behind the voltage E_{AO} by 60 degrees and therefore falls on top of E_{CO} reversed. E_{CO} reversed is 30 degrees out of phase with E_{AC}. From Eq. (12) we find that 30 degrees $-$ 60 degrees $= -30$ degrees, and since the cosine of -30 degrees is positive the meter will read positive. This is also shown by the fact that the component of I_{aA} that is in phase with the voltage E_{AC} is in the same direction as the voltage. The in-phase component is shown as *oy*. It will be noticed that I_{bB} lags behind E_{BC} by 90 degrees under these conditions. This corresponds to 30 degrees + 60 degrees = 90 degrees. This meter stands still, since there is no in-phase component whatever. The same reasoning holds for Fig. 42*b*. In Fig. 42*c* θ is about 75 degrees, and 30 degrees − 75 degrees = −45 degrees. Again, since the cosine of −45 degrees is positive, this meter will read positive. This is also shown by the fact that the in-phase component *oy* of I_{aA} is in the same direction as E_{AC}. The case of the current I_{bB}, however, is different. The component *ox* of this current that is in phase with the voltage E_{BC} is opposite in direction, thereby causing the meter to read backward. Also from Eq. (13) we have the cosine of 30 degrees + 75 degrees = 105 degrees. This angle is in the second quadrant and therefore its cosine is negative. The same reasoning holds for Fig. 42*d*.

34. Determination of Whether a Wattmeter Reading Is Positive or Negative.

If two wattmeters are connected to a three-phase circuit as in Fig. 43*a*, and it was not known whether the power factor was greater or less than 50 per cent, we should not know whether the smaller reading instrument should be considered positive or negative. We may experimentally determine whether the smaller reading wattmeter should be subtracted or added in several different ways. If the meters are identical and it is known which connections produce positive readings, tracing the connections will generally determine whether the smaller reading is positive or negative. If twisted-cord potential leads are used, this is sometimes difficult to accomplish.

Another method consists in increasing the power factor by increasing the load on an induction motor or adding additional resistance to the load. If, when this is done, both wattmeters increase their readings, they should have both been considered positive. If,

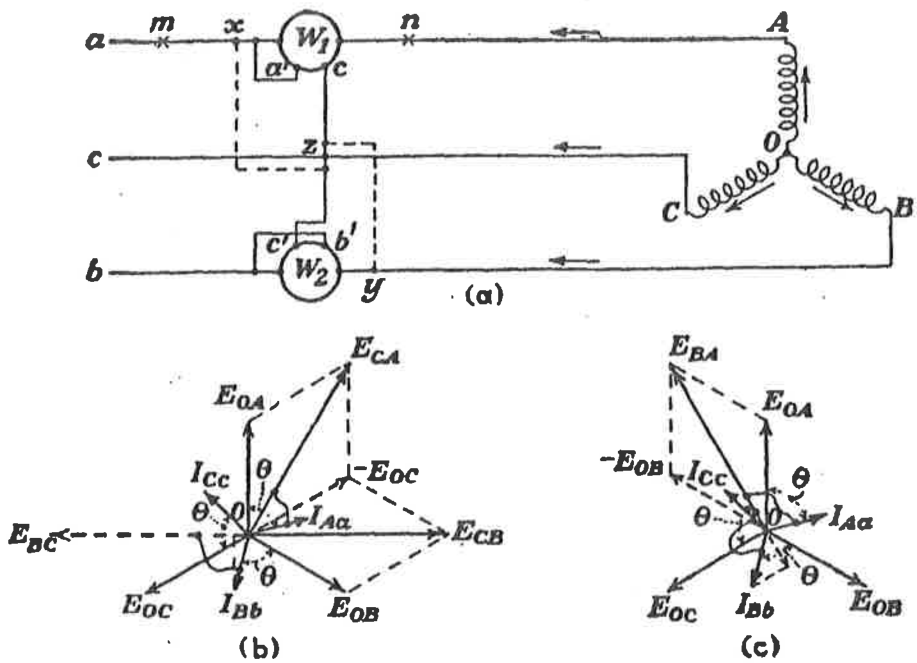

FIG. 43.—Test to determine whether a wattmeter reading is negative or positive.

however, the smaller reading instrument decreases or reverses, it should have been considered negative.

Isolating the larger reading wattmeter by opening the circuit, say at *m* and *n* (Fig. 43*a*), is a third method. When this is done, the circuit reduces from three phase to single phase. If the smaller reading wattmeter after the change continues to read forward, its reading was originally positive. If it reverses after the change, its reading was originally negative.

Finally, assuming that both meters are connected to read forward, we may shift the common potential lead z of either wattmeter to the other line wire at x or y (Fig. 43*a*). If either wattmeter reverses after the change, the power factor was less than 50 per cent and the smaller reading meter should be considered negative. The proof of this fact is given in Figs. 43*b* and *c*. We have assumed a power factor less than 50 per cent and currents flowing away from the load toward the source. This is opposite to the assumption made in Figs. 41 and 42, but, as was there stated, either assumption may be made. In Fig. 43*b* we draw E_{OA}, E_{OB}, and E_{OC} to correspond to our assumption of current flow, and I_{Aa}, I_{Bb}, and I_{Cc} lagging behind their respective coil voltages by an angle greater than 60 degrees. Since we have assumed current flowing out of the load, it will pass through W_1 from right to left. If this meter is to read positive, the voltage must act from right to left. Such a voltage is therefore E_{CA}, found by adding E_{OC} reversed to E_{OA}. In order for W_2 to read positive, its potential leads must be interchanged, but supposing that we were unable to see that the potential leads were interchanged and therefore do not know that W_2 should be considered negative. The normal connection for W_2 with current I_{Bb} flowing from right to left is with voltage E_{CB} also flowing from right to left. However, since the potential leads are interchanged, this meter has voltages E_{BC} and I_{Bb} applied to it. Our diagram actually shows that this would give a positive reading, since the in-phase component of I_{Bb} is in the same direction as E_{BC}. Suppose now that we shift the potential lead z of W_1 to y. Since y is the entering potential lead for W_1, according to our present current assumption, the voltage now applied to W_1 is E_{BA}. E_{BA} is shown in diagram *c* of Fig. 43. It will be noticed that the angle between I_{Aa} and E_{BA} is greater than 90 degrees, therefore W_1 will reverse. This proves that W_2 should have been considered negative. We might have chosen to shift the potential lead *z* of W_2 to *x*. Since, again, our entering potential lead for W_2 is on the right, this will apply voltage E_{BA} to W_2. From diagram *c* of Fig. 43 we see that the angle between I_{Bb} and E_{BA} is greater than 90 degrees, and therefore W_2 will reverse. This indicates, as in the case of W_1, that W_2 should have been considered negative.

The above proof is not included here because it, itself, is of any particular importance but to illustrate the statement made previously that currents may be assumed either flowing toward or away from the load in analyzing polyphase meter connections. Also it illustrates the entering and leaving potential and current-terminal idea. This idea is extremely useful when used in connection with vector diagrams.

Checking the correctness of polyphase wattmeter or watt-hour meter connections is a subject in itself. For more advanced study of this subject the reader is referred to a 1927-1928 serial report of the National Electrical Light Association Meter Committee[1] and the author's *Bulletin* 8.[2]

[1] Methods of Determining Correctness of Watthour Meter Connections, *Publication* 278-31, serial report of Meter Committee, N.E.L.A., New York, March, 1928.

[2] "Connection Checks for Polyphase Watthour Meters," Engineering Extension Service, Purdue University, Lafayette, Ind.

35. Importance of Phase Rotation.

As stated in Art. 31, the effect of a change in phase rotation is to interchange Eqs. (11) and (12) between the two wattmeters. If the meters are properly connected, the final result (Eq. (13)) will be the same. However, if the meter is improperly connected, a change in phase rotation may alter the final result. Just looking at a meter connection or a three-phase circuit will not tell us the phase rotation, and if we are truly to represent the actual conditions existing by a vector diagram, we must know the phase rotation before we start. It is a relatively simple matter to test for phase rotation. The test is made with two lamps of equal size and a high inductive reactance connected in Y. These, in turn, are connected to the three line wires of a three-phase system, as shown in Figs. 44*a* and *b*. Ordinarily, incandescent lamps may be used and a potential coil from a watthour meter. When this connection is made it will be found that one lamp is appreciably brighter than the other. The sequence is then reactance lead, dim lead, and bright lead. To prove this statement we shall use the indirect method, *i.e.*, we shall assume the statement correct and then show that the results substantiate this assumption. It is obvious that the load consisting of the two lamps and reactance coil is not balanced. The currents drawn by the lamps will be in phase with their respective coil voltages, since a lamp is a unity power-factor load (see Rule *a*, Chap. II, Art. 17). On the other hand, the current drawn by the reactance coil will lag behind its phase voltage (see Rule *b*, Chap. II, Art. 17). Nevertheless, the vector sum of the three currents must equal zero (see Rule 2, Chap.

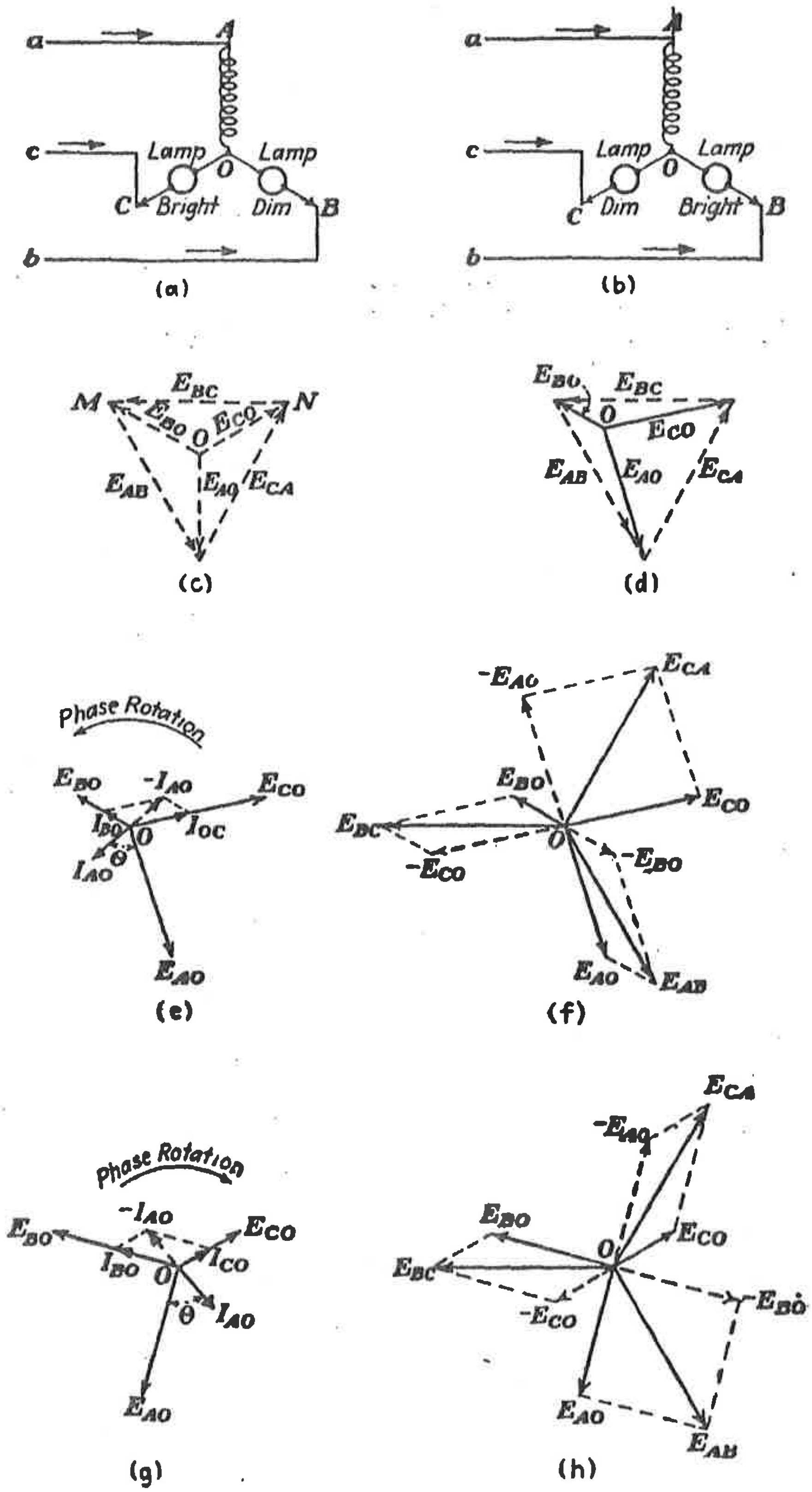

FIG. 44.—Test for phase rotation.

V, Art. 29). Besides these three conditions there are three more imposed on our circuit. The line voltages are equal and symmetrical, that is, 120 degrees apart. This is true because a relatively small load connected to the system cannot change the system voltages appreciably. Also the difference of the coil voltages taken in pairs must produce these same line voltages (see Fig. 38*b*). Finally, if the lamps are equal in size and one is to be brighter than the other, the drop across the lamps cannot be equal. One must be larger than the other. This means that the coil voltages are not equal. We shall now build a vector diagram which will satisfy all of these conditions. In Fig. 44*a*, assume currents flowing toward the load. For the moment assume a balanced load. Then the vector diagram Fig. 44*c* would be obtained. The three coil voltages E_{AO}, E_{BO}, and E_{CO} are equal and 120 degrees apart. The three line voltages are found by the triangular method of subtraction. Thus:

$$\dot{E}_{AB} = \dot{E}_{AO} - \dot{E}_{BO},$$
$$\dot{E}_{BC} = \dot{E}_{BO} - \dot{E}_{CO},$$

and

$$\dot{E}_{CA} = \dot{E}_{CO} - \dot{E}_{AO}.$$

This cannot be the actual situation in our circuit, for, if it were, the lamps would both be of the same degree of brightness. The voltage E_{BO} must be less than E_{CO} and probably less than E_{AO}. The only way in which this can occur and still satisfy the vector equations above is for the point O to shift in the general direction of the vertex M. It might shift along the line OM or it might shift above or below this line. It must, however, shift toward M. Just where it shifts depends upon the relative magnitudes of the currents drawn by the three phases. Figure 44*d* shows the phase voltages after the shift or the true phase voltages that must exist. It will be noticed that by the triangular method the differences $\dot{E}_{AO} - \dot{E}_{BO}$, $\dot{E}_{BO} - \dot{E}_{CO}$, and $\dot{E}_{CO} - \dot{E}_{AO}$ still produce, respectively, the same identical line voltages E_{AB}, E_{BC}, and E_{CA}. Now let us draw in the currents. For the sake of clearness the three-phase voltages obtained in diagram *d* are reproduced in *e*. Since E_{BO} is smaller than E_{CO}, I_{BO} is smaller than I_{CO}, but both are in phase with their respective voltages. Since the vector sum of the three-phase currents must be zero, we obtain I_{AO} by reversing the vector sum of I_{BO} and I_{CO}. I_{AO} is out of phase with its voltage E_{AO} by the angle Θ. Since the current I_{AO} flows through a reactance, it must lag its voltage. Therefore the angle Θ is a lagging angle and the phase rotation must be counterclockwise. The phases are then in the order *ABC*. Figure 44*f* shows the line voltages obtained by the parallelogram method. Since phase rotation was proved counterclockwise, the line-voltage sequence is $E_{AB} - E_{BC} - E_{CA}$.

Now consider Fig. 44*b*. In this case the bright lamp is found in the phase that contained the dim lamp in Fig. 44*a*. From what has just been said, it is obvious that the voltage E_{CO} must be smaller than E_{BO}. This means that the shift of the neutral point *O* must be in the general direction of the vertex *N*. The three coil voltages will then be as shown in Fig. 44*g*. The currents are drawn as before and I_{AO} determined by reversing the vector sum of I_{BO} and I_{CO}. I_{AO} is again out of phase with its voltage by the angle Θ, but with this difference: It falls to the right of E_{AO}. However, we know that θ is a lagging angle. In order to make it a lagging angle, the phase rotation must be clockwise. The phases then count *ACB*. Figure 44*h* shows the combination of the coil voltages which produce the same line voltages as in Fig. 44*f*. Since the phase rotation is clockwise, the voltage sequence is $E_{AB} - E_{CA} - E_{BC}$.

It is true that the above proof, in itself, is of little practical importance. The benefit to be gained from its study is the application of the vector and the method of analysis employed. Polyphase systems, whether balanced or unbalanced, have certain fundamental restrictions placed upon them which must not be violated. Practically all of them are well illustrated in this problem, and the reader will not find his time wasted who gives it a careful study.

36. Examples of Improper Connections in Three-Phase Three-wire Metering.

Problems arising in the two-wattmeter method of measuring power in a three-phase, three-wire system or in measuring energy by a polyphase watthour meter usually have to do with improper connections or a modified connection requiring the use of a multiplying factor. Some one discovers an improperly connected meter and then wishes to calculate what the meter actually recorded. Sometimes it is possible to determine a multiplying factor which applies under all conditions, but usually such calculations result in determining that the registration of the meter is a function of the power factor. That is, it reads different for different power factors. The following examples will suffice to illustrate these two cases.

Consider this problem. It is desired to meter a 220-volt load from a delta-connected bank of transformers having the customary double-coil, low-voltage winding. Only a 110-volt polyphase watthour meter is available. Some one suggests that the 110-volt meter can be used with its potential coils connected to the midpoints of the low-tension windings and a

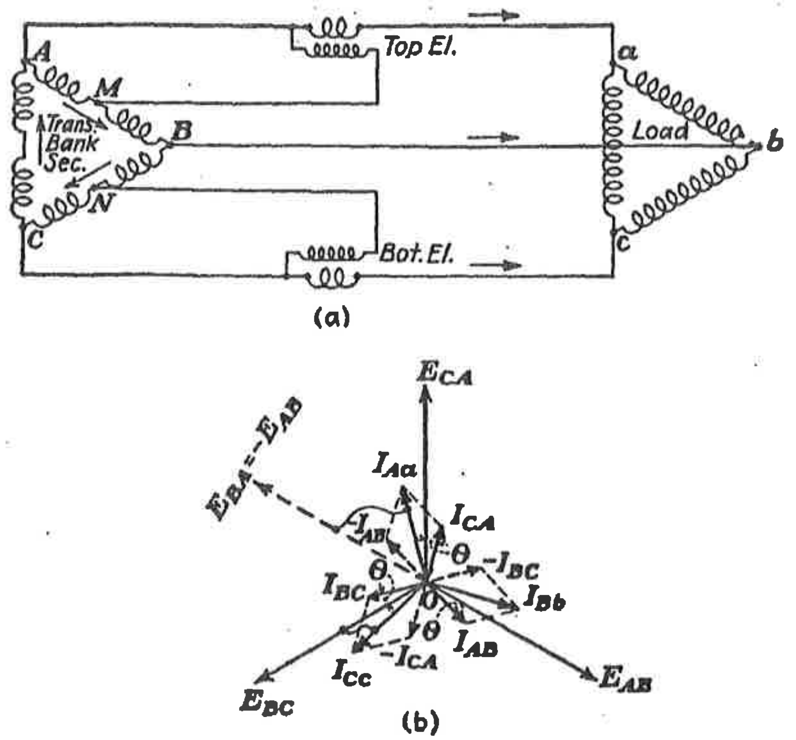

FIG. 45.—Special connection for a polyphase meter.

multiplying factor of two. Let us see if he is correct. The meter is connected as in Fig. 45*a*.

Assuming currents flowing as shown, the entering potential and current terminals are on the left side of the meter. In Fig. 45*b* the voltages E_{CA}, E_{AB}, and E_{BC} are drawn according to this assumption to represent the voltages of the transformer bank. The phase currents I_{CA}, I_{AB}, and I_{BC} are then drawn lagging behind their respective voltages by θ degrees. The line currents I_{Aa} and I_{Cc}, which pass through the meter coils, are then found from the vector equations

$$\dot{I}_{Aa} = \dot{I}_{CA} - \dot{I}_{AB}$$

and

$$\dot{I}_{Cc} = \dot{I}_{BC} - \dot{I}_{CA}.$$

The top element of the meter has the current I_{Aa} flowing from left to right and the voltage $AM = \frac{1}{2}AB$ flowing from right to left. The voltage $\frac{1}{2}AB$ acting from right to left is the same as the voltage $\frac{1}{2}BA$ acting from left to right. This latter voltage is the one we must consider, since it is acting in the same direction as the current we are considering. From our diagram we see that the angle between I_{Aa} and E_{BA} is $(30 + \theta)$. Therefore the top element records $\frac{1}{2}E_{AB}I_{Aa} \cos (30 + \theta)$. The bottom element has the current I_{Cc} and the voltage $NC = \frac{1}{2}BC$ both flowing from left to right. Since they are both flowing in the same direction through the meter, it is not necessary to reverse one of them. The angle between $\frac{1}{2}E_{BC}$ and I_{Cc} is $(30 - \theta)$ degrees. Consequently the bottom element reads $\frac{1}{2}E_{BC}I_{Cc} \cos (30 - \theta)$.

It will be noticed that the two equations just determined are equal to ½ Eqs. (11) and (12). Therefore the meter records just half of what it should and a multiplying factor of two must be used. As the angles are not affected in any way, this multiplying factor will hold for any power factor. The next problem illustrates a case where this is not true.

Suppose a polyphase watthour meter was found connected as shown in Fig. 46*a* and it is desired to determine what it had been recording. A phase-rotation test gives the phase sequence as *ABC*. It is obvious from the connections that the bottom element has its potential coil reversed. Figure 46*b* shows the vector diagram corresponding to the connection shown in Fig. 46*a*. Assuming currents flowing toward the load, our entering potential and current terminals are on the left-hand side of the meter. Consequently, voltage E_{AC} is applied to the top element and E_{CB} to the bottom element. Therefore the top element reads $E_{AC}I_{aA}\cos(30 - \theta)$. This is Eq. (11), and therefore

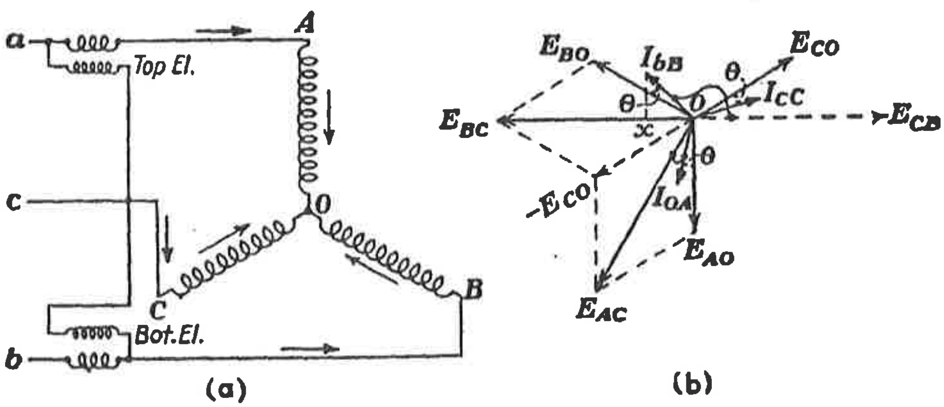

FIG. 46.—Polyphase watthour meter with one potential coil reversed.

the top element is reading correctly. The bottom element has applied to it the current I_{bB} and the voltage E_{CB}. The angle between E_{CB} and E_{BO} is 150 degrees. Therefore the angle between E_{CB} and I_{bB} is $(150 - \theta)$ and the bottom element reads $E_{CB}I_{bB}\cos(150 - \theta)$. This is not Eq. (12), and therefore the bottom element is not reading correctly. The meter as a whole reads

$$P = E_{\text{line}}I_{\text{line}}[\cos(30 - \theta) + \cos(150 - \theta)].$$

Using the expansion formulas of the Appendix and Table III, we get

$$P = E_{\text{line}} I_{\text{line}} (\cos 30 \cos \theta + \sin 30 \sin \theta + \cos 150 \cos \theta + \sin 150 \sin \theta)$$

$$= E_{\text{line}} I_{\text{line}} \left(\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta \right)$$

$$= E_{\text{line}} I_{\text{line}} \sin \theta. \tag{14}$$

Equation 14 is not the power equation for a three-phase circuit. At unity power factor $\sin \theta = 0$ and the meter stands still. At other values of θ the meter runs forward. It will run at the fastest rate when $\theta = 90$ degrees. Therefore the registration depends upon the power factor, and unless we can determine what the power factor was and that it was constant, we are powerless to determine what the meter was recording.

CHAPTER VII

POWER OR ENERGY MEASUREMENTS IN A THREE-PHASE, FOUR-WIRE SYSTEM

37. Number of Wattmeters Required.

To measure power or energy correctly in a three-phase, four-wire system under all conditions requires three wattmeters or a three-element polyphase watthour meter. This fact is now pretty generally conceded. In the past, however, several schemes have been used which are more or less satisfactory under conditions generally met in practice. For a complete discussion see the author's *Bulletin* 13.[1] One problem on this subject will now be considered.

38. Two Element Polyphase Wattmeter Method.

One method of measuring three-phase, four-wire energy consists in using the standard two-element polyphase watthour meter with three current transformers connected in delta and the potential leads connected from line to neutral as illustrated in Fig. 47*a*. A certain individual did not have potential transformers of the proper ratio and wanted to know if the connection shown in Fig. 47*b* would work. For the sake of simplicity the potential transformers have been omitted. Before we can determine whether or not the

[1] "Comparative Accuracy of Three-phase, Four-wire Metering Methods," Engineering Extension Department, Purdue University, Lafayette, Ind.

connections of Fig. 47*b* are satisfactory, it will be necessary to study the correct method Fig. 47*a*. Here, for the first time, we have introduced polarity marks. These are markers placed on instrument transformers

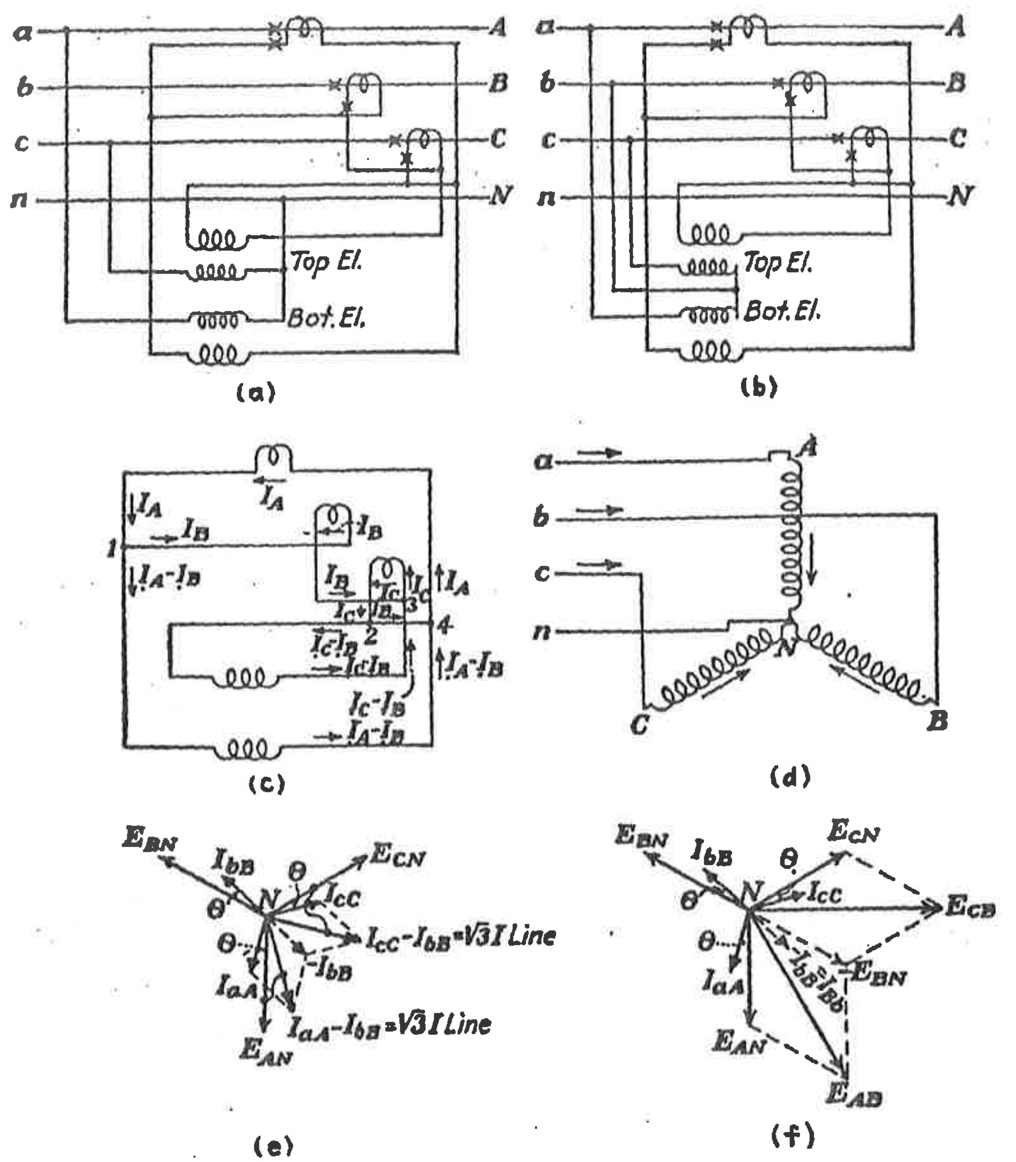

FIG. 47.—Three-phase, four-wire metering.

by the manufacturer to indicate how the transformers are wound. They are so placed that if the current enters a polarity mark on the line side it simultaneously flows out of a polarity mark on the meter side. Thus in Fig. 47*a*, if current is flowing from *a* to *A* in the line, it flows the opposite way on the meter side. Therefore if the transformer were removed and the line connected through to the meter, there would be no change in the direction of the meter current. In Fig. 47*d* is shown the load that is connected to either Fig. 47*a* or Fig. 47*b*. Currents are assumed flowing toward the load. Consequently, they flow the other way in the secondaries of the current transformers. However, it is really the line currents with which we deal in our vector diagrams, as we shall see presently. Before drawing the vector diagram it is necessary to determine just which currents are actually flowing in our meter coils. As an aid to determining the actual currents in the current coils of the meter, we have reproduced the current system of Fig. 47*a* in Fig. 47*c*. Current I_{aA} (abbreviated I_A) flows down into the junction 1. Current I_{bB} (abbreviated I_B) flows out of junction 1 to the right. Therefore current $\dot{I}_A - \dot{I}_B$ flows out of junction 1 toward the bottom-element current coil. This current is a vector difference. At junction 2 current I_{cC} (abbreviated I_C) flows down into the junction. I_B flows out of junction 2 to the right. Therefore current $\dot{I}_C - \dot{I}_B$ flows out of junction 2 toward the top-element current coil. This current is also a vector difference. At junction 3, I_B comes in from the left, I_C goes out at the top, and $\dot{I}_C - \dot{I}_B$ comes in from the bottom. At junction 4, I_B comes in from the left, I_A goes out at the top, and $\dot{I}_A - \dot{I}_B$ comes in from the bottom. The path followed by $\dot{I}_B$ is then from its current transformer to 3, through the top element to 2, to 4, through the bottom element to 1, and back to its transformer.

The fact that the vector sum of the currents at a junction point must be zero enables us to check the flow of currents just determined. Assume that the currents flowing into a junction are positive and those flowing out are negative. Then, at junction 1,

$$0 = \dot{I}_A - \dot{I}_B - (\dot{I}_A - \dot{I}_B) = \dot{I}_A - \dot{I}_B - \dot{I}_A + \dot{I}_B = 0,$$
which checks.

At 2,

$$0 = \dot{I}_C - \dot{I}_B - (\dot{I}_C - \dot{I}_B) = \dot{I}_C - \dot{I}_B - \dot{I}_C + \dot{I}_B = 0,$$
which checks.

At 3,

$$0 = \dot{I}_B + (\dot{I}_C - \dot{I}_B) - \dot{I}_C = \dot{I}_B + \dot{I}_C - \dot{I}_B - \dot{I}_C = 0,$$
which checks.

At 4,

$$0 = (\dot{I}_A - \dot{I}_B) + \dot{I}_B - \dot{I}_A = \dot{I}_A - \dot{I}_B + \dot{I}_B - \dot{I}_A = 0,$$
which checks.

Now that we know the currents that flow in the current coils of our meter, we can draw the vector diagram. This is done in Fig. 47*e*. E_{AN}, E_{BN}, and E_{CN} are drawn to correspond to the current flow assumed in Fig. 47*d*. The currents are assumed lagging by the angle θ. The meter currents $\dot{I}_{aA} - \dot{I}_{bB}$ and $\dot{I}_{cC} - \dot{I}_{bB}$ are then found by adding I_{bB} reversed to I_{aA} and I_{cC} in turn. For balanced currents the magnitude of the resultant is equal to $\sqrt{3}$ times the line current (see Art. 30). The resultant current in the bottom element is associated with the voltage E_{AN} and the angle between them is $(30 - \theta)$. The resultant current in the top element is associated with the voltage E_{CN} and the angle between them is $(30 + \theta)$. Therefore the top element records

$E_{CN}\sqrt{3}I_{cC}$ cos $(30 + \theta)$ and the bottom element $E_{AN}\sqrt{3}I_{aA}$ cos $(30 - \theta)$. The meter as a whole records

$$\begin{aligned} P &= \sqrt{3}E_{coil}I_{coil} \cos (30 + \theta) + \cos (30 - \theta) \\ &= \sqrt{3}E_{coil}I_{coil} (\cos 30 \cos \theta - \sin 30 \sin \theta + \cos 30 \cos \theta + \sin 30 \sin \theta) \\ &= \sqrt{3}E_{coil}I_{coil}\left(2\frac{\sqrt{3}}{2} \cos \theta\right) \\ &= 3E_{coil}I_{coil} \cos \theta_{coil}. \qquad (15) \end{aligned}$$

Equation (15) states that the total power is equal to three times the power per phase, which is correct for a balanced load.

If the load is unbalanced, the method is still correct provided the voltages remain symmetrical. This may be shown by taking the currents in the coils separately. It was shown in Fig. 47*c* that the current I_{bB} circulates backward through both current coils. This is the same as $I_{bB} = I_{Bb}$ circulating forward. I_{Bb} makes an angle $(60 + \theta_B)$ with E_{CN}, and $(60 - \theta_B)$ with E_{AN}. Therefore the top element records $E_{CN}I_{cC}$ cos $\theta_c + E_{CN}I_{Bb} \cos (60 + \theta_B)$, and the bottom element $E_{AN}I_{aA} \cos \theta_A + E_{AN}I_{Bb} \cos (60 + \theta_B)$. The meter as a whole records $E_{CN}I_{cC} \cos \theta_C + E_{AN}I_{aA} \cos \theta_A + I_{Bb} [E_{CN} \cos (60 + \theta_B) + E_{AN} \cos (60 - \theta_B)]$.

Numerically,

$$E_{AN} = E_{CN} = E_{BN} \text{ and } I_{Bb} = I_{bB}.$$

Hence the total reading becomes $E_{CN}I_{cC} \cos \theta_C + E_{AN}I_{aA} \cos \theta_A + E_{BN}I_{bB} [\cos (60 + \theta_B) + \cos (60 - \theta_B)]$. This will reduce to

$$E_{CN}I_{cC}\cos\theta_C + E_{AN}I_{aA}\cos\theta_A + E_{BN}I_{bB}(\cos 60\cos\theta_B - \sin 60\sin\theta_B + \cos 60\cos\theta_B + \sin 60\sin\theta_B)$$
$$= E_{CN}I_{cC}\cos\theta_C + E_{AN}I_{aA}\cos\theta_A + E_{BN}I_{bB}\cos\theta_B. \quad (16)$$

Thus the currents may be different and the angles different but the voltages must be equal and differ in phase by 120 degrees, or E_{AN} will not equal E_{BN} and E_{CN} will not equal E_{BN}.

Returning now to the connection of Fig. 47*b*. Here the voltages are connected from line to line instead of line to neutral. The current system, however, is unchanged. The top element has voltage E_{CB} applied to it. This voltage is found by adding E_{BN} reversed to E_{CN}. The bottom element has voltage E_{AB} applied to it. This voltage is found by adding E_{BN} reversed to E_{AN}. The top element then records $E_{CB}I_{cC}\cos(30 - \theta_C) + E_{CB}I_{Bb}\cos(30 + \theta_B)$. In like manner, the bottom element reads $E_{AB}I_{aA}\cos(30 + \theta_A) + E_{AB}I_{Bb}\cos(30 - \theta_B)$. Now under the assumption of symmetrical voltages, E_{CB} and E_{AB} are each equal to the $\sqrt{3}$ times a coil voltage (see Art. 29), and $I_{Bb} = I_{bB}$ numerically. Expanding the angles and substituting coil voltage for line voltage, we obtain the following four expressions:

$$\sqrt{3}E_{\text{coil}}I_{cC}(\cos 30\cos\theta_C + \sin 30\sin\theta_C).$$
$$\sqrt{3}E_{\text{coil}}I_{bB}(\cos 30\cos\theta_B - \sin 30\sin\theta_B).$$
$$\sqrt{3}E_{\text{coil}}I_{aA}(\cos 30\cos\theta_A - \sin 30\sin\theta_A).$$
$$\sqrt{3}E_{\text{coil}}I_{bB}(\cos 30\cos\theta_B + \sin 30\sin\theta_B).$$

It is obvious that these expressions will not reduce to Eq. (16) if the currents and angles are different.

If the system is balanced, then the above four expressions will combine as follows:

$$\text{Total power} = \sqrt{3}E_{\text{coil}}I_{\text{coil}}\, 4 \times \frac{\sqrt{3}}{2} \cos \theta_{\text{coil}}$$

$$= 2 \times 3E_{\text{coil}}I_{\text{coil}} \cos \theta_{\text{coil}}.$$

This expression is twice Eq. (15). Therefore at balanced loads, and at any power factor, the meter reads double what it should, and a multiplying factor of ½ should be used. On unbalanced loads, about all that can be said is that the meter records the four expressions given above taken collectively and that it should record Eq. (16).

CHAPTER VIII

REACTIVE VOLT-AMPERE MEASUREMENT

39. Principles of Reactive Volt-ampere Metering.

The measurement of reactive power or, better, reactive volt-amperes is of increasing importance due to the increased use of power-factor clauses in rate schedules.[1] So far we have dealt entirely with power or energy measurements. In all methods for measuring power or energy we had to combine in our metering device the proper voltage or voltages with the proper in-phase component of some current or currents. Reactive volt-ampere measuring schemes seek to combine certain voltages with the other or quadrature component of the current. Thus in Fig. 48*a* the current I_A is resolved into two components $OB = I \cos \theta$ in phase with the voltage and $BA = I \sin \theta$ in quadrature or 90 degrees with the voltage. The power delivered is

$$P = EI \cos \theta \qquad (5)$$

and the reactive volt-amperes

$$Q = EI \sin \theta. \qquad (17)$$

For the three-phase circuit

$$Q = \sqrt{3}EI \sin \theta. \qquad (18)$$

If the sides of the triangle OBA are each multiplied by the voltage, we obtain the triangle of Fig. 48*b*.

[1] See Part IV, Chap. IX.

In this triangle the base is the power in watts, the altitude the reactive volt-amperes, and the hypotenuse the volt-amperes. Dividing by 1,000 we obtain, respectively, kilowatts, reactive kilovolt-amperes, and

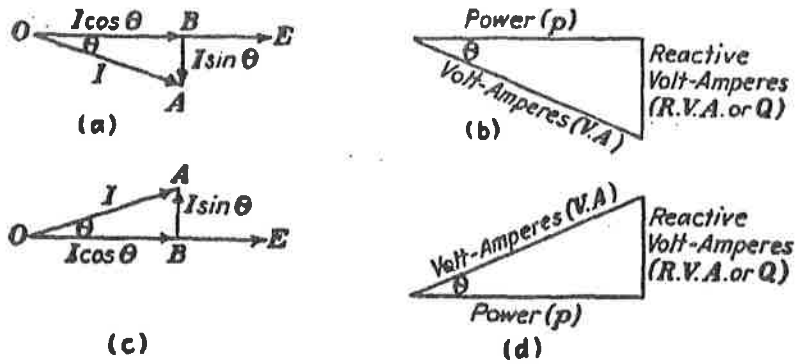

FIG. 48.—Relation of power to reactive volt-amperes.

kilovolt-amperes. The relation of these three quantities is obtained from this triangle as follows:

$$\text{Kv.-a.} = \sqrt{\text{kw.}^2 + (\text{r kv.-a.})^2}. \qquad (19)$$

$$\text{Power factor} = \frac{\text{kw.}}{\text{kv.-a.}}.^{1} \qquad (20)$$

If the current is leading the voltage instead of lagging behind the voltage, as in Fig. 48*c*, a similar triangle is obtained, as in Fig. 48*d*. The relations as expressed in Eqs. (19) and (20) are the same. Also it should be noticed that Eqs. (19) and (20) hold for unbalanced as well as balanced three-phase circuits. In the case of unbalanced systems, the power factor has no other significance than the ratio of the kilowatts to the kilovolt-amperes. It does not equal the cosine of any particular angle. The usual commercial method of measuring power factor for rate purposes employs two meters, one measuring kilowatt-hours and the other reactive kilovolt-ampere-hours.

[1] See also Part IV, Chap. X, Art. 48.

From these two readings the kilovolt-ampere-hours are calculated from Eq. (19) and the power factor from (20). Some manufacturers mount the two meters in the same case and combine their registrations mechanically or graphically to give the kilovolt-amperes or the power factor or both directly.

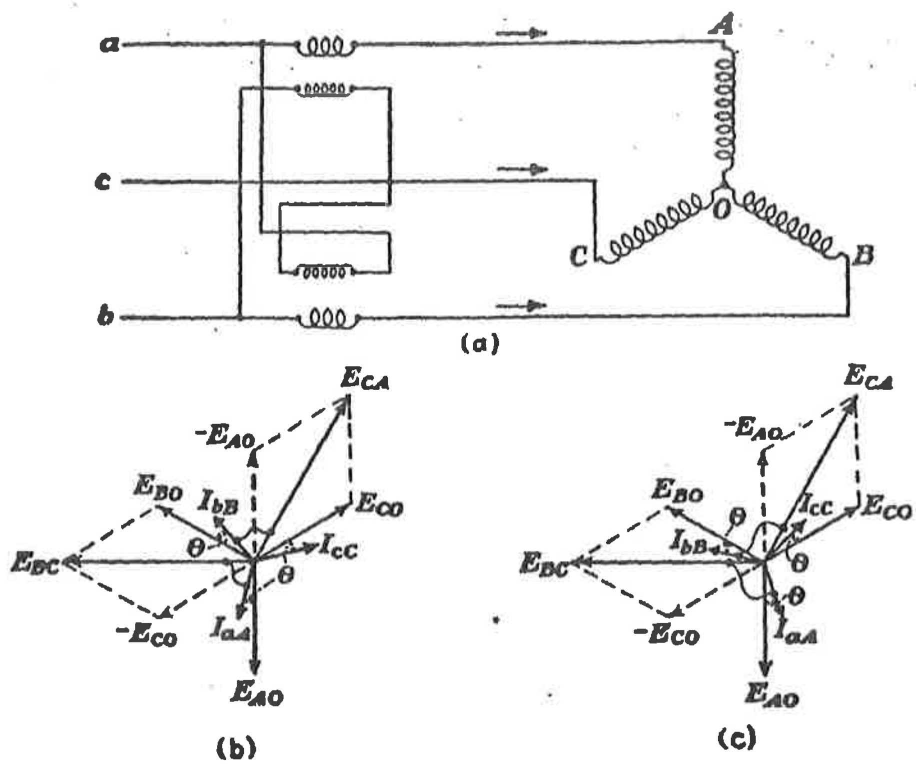

FIG. 49.—Reactive volt-ampere measurement.

The fundamental principle of reactive kilovolt-ampere measurement consists in combining the currents in the current coils of the meter with voltages which at unity power factor are at right angles to these currents. This is generally accomplished by cross-phasing the potential leads. It is based upon the assumption that the voltages remain balanced and symmetrical. This assumption is generally sufficiently correct in practice to warrant its use. The basic connection for measuring reactive kilovolt-amperes in a three-phase, three-wire system is illustrated in Fig. 49*a*.

It will be noticed that the potential that was normally on the top element is placed on the bottom element but reversed. The potential that is normally on the bottom element is placed on the top element and in the same direction.

Assuming current flow toward the load, we establish our entering potential and current terminals on the left side of the meter. Therefore the top element has current I_{aA} and potential E_{BC} applied to it. The bottom element has current I_{bB} and potential E_{CA} applied to it. To show this vectorially for lagging currents, we construct Fig. 49*b*. E_{AO}, E_{BO}, and E_{CO} are drawn to represent our load-coil voltages according to our current-flow assumption. E_{BC} is then found by reversing E_{CO} and adding it to E_{BO}. E_{CA} is found by reversing E_{AO} and adding it to E_{CO}. It should be noticed that E_{BC} is at right angles to E_{AO}, the unity power-factor position of current I_{aA}; and that E_{CA} is at right angles to E_{BO}, the unity power factor position of I_{bB}. This is the condition that must be met in reactive kilovolt-ampere metering. On the other hand, unless the voltages remain fixed in position and magnitude, this 90-degree relation will vary and thus introduce an error. In practice, the voltages usually do remain fixed within small limits, so that the error introduced is not generally commercially serious.

The angle between E_{BC} and I_{aA} is $(90 - \theta)$ and the angle between E_{CA} and I_{bB} is $(90 - \theta)$. Therefore the top element records $E_{BC}I_{aA} \cos (90 - \theta)$ and the bottom $E_{CA}I_{bB} \cos (90 - \theta)$. In a balanced system the meter as a whole will record $2E_{\text{line}}I_{\text{line}}$ (cos 90 $- \theta$). Expanding we get

$$2E_{\text{line}}I_{\text{line}}\,(\cos 90 \cos \theta + \sin 90 \sin \theta) = 2E_{\text{line}}I_{\text{line}}\,(0 \times \cos \theta + 1 \times \sin \theta)$$
$$= 2E_{\text{line}}I_{\text{line}} \sin \theta. \qquad (21)$$

Equation (21) is not Eq. (18), therefore a multiplying factor of $\frac{\sqrt{3}}{2} = 0.866$ must be used to reduce Eq. (21) to (18).

We learned in Chap. VI, Art. 31, that it made no difference as to the net result whether the currents were leading or lagging so far as power or energy measurements were concerned. In the case of reactive volt-ampere measurement, however, reversal of the watthour meter as a whole takes place. Thus in Fig. 49*c* we have represented leading currents for the load and meter connections of Fig. 49*a*. By inspection we see that the angle between I_{aA} and E_{BC} is greater than 90 degrees, and therefore this element will rotate backward. Likewise the angle between I_{bB} and E_{CA} is also greater than 90 degrees so that this element rotates backward. Since both elements produce reversed torque, the meter as a whole will rotate backward on leading currents. The speed at which it rotates may be found as follows:

$$W_1 = E_{BC}I_{aA} \cos (90 + \theta).$$
$$W_2 = E_{CA}I_{bB} \cos (90 + \theta).$$

On a balanced symmetrical system,

$$W_1 + W_2 = E_{\text{line}}I_{\text{line}}\,(\cos 90 \cos \theta - \sin 90 \sin \theta + \cos 90 \cos \theta - \sin 90 \sin \theta)$$
$$= E_{\text{line}}I_{\text{line}}\,(-2 \sin \theta)$$
$$= -2E_{\text{line}}I_{\text{line}} \sin \theta. \qquad (22)$$

Equation (22) is equal to but opposite in sign to Eq. (21). Therefore for the same leading power-factor angle, the meter will rotate backward at the same speed that it rotates forward on lagging power factor. A study of the diagrams in Fig. 48 would indicate that this is as it should be, since the reactive component of the current in the case of the leading current is opposite but equal to the reactive component of current for the same angle of lagging power factor. In practice the rate schedule is sometimes such that the customer is not allowed to overexcite his synchronous motors and thereby run his reactive meter backward. To prevent such a possibility, the reactive meter is equipped with a ratchet to prevent backward rotation but allowing forward rotation.

40. Use of the Phasing Transformer.

Because of the difficulty of making a non-technical customer understand its purpose, a multiplying factor is not generally used in practice. The alternative of an odd register ratio or watthour constant is equally undesirable. The usual procedure is to use a phasing transformer. The connections are illustrated in Fig. 50*a*. Two autotransformers *umvp* and *ulwn* are connected to the line wires as shown. Transformer *umvp* is wound so that $up = 1.154\ uv$ and m is a midpoint of up. In like manner $un = 1.154\ uw$ and l is a midpoint of un. The line voltages are applied between vu and wu. Thus the voltage

$$up = 1.154E_{\text{line}}$$

and voltage

$$un = 1.154E_{\text{line}}.$$

Also since m and l are midpoints,

voltage,

$$um = mp = ul = ln = 0.577E_{\text{line}}.$$

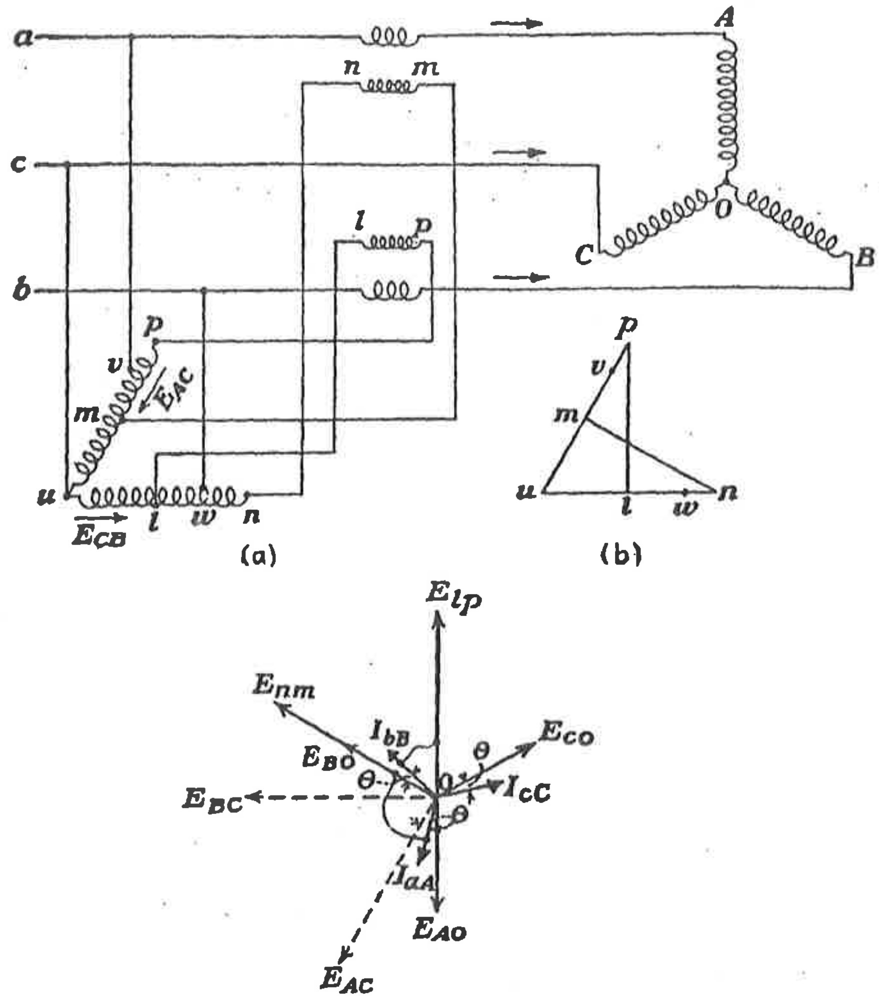

Fig. 50.—Phasing transformer for reactive kilovolt-ampere measurement.

From Fig. 50*b* we see that

$$pl = mn = \sqrt{\overline{up}^2 - \overline{ul}^2} \text{ or } pl = mn = \sqrt{\overline{1.154E_{\text{line}}}^2 - \overline{0.577E_{\text{line}}}^2}$$

$$= \sqrt{1.333E^2_{\text{line}} - 0.333E^2_{\text{line}}}$$

$$= \sqrt{E^2_{\text{line}}}$$

$$= E_{\text{line}}.$$

The top element has the voltage *nm* applied to it. The voltage *nm* is equal in magnitude to the line voltage E_{AC} but it is at right angles to E_{AC}. The bottom element has the voltage *lp* applied to it. This is equal in magnitude to the voltage E_{BC} but it is at right angles to E_{BC}. From Fig. 50*c* we see that the top element has applied to it $I_{aA}E_{nm} \cos (120 - \theta)$ and the bottom element $I_{bB}E_{lp} \cos (60 - \theta)$. Expanding and adding, we get the reactive volt-amperes

$$Q = E_{nm}I_{aA} (\cos 120 \cos \theta + \sin 120 \sin \theta) + E_{lp}I_{bB} (\cos 60 \cos \theta + \sin 60 \sin \theta)$$

$$= E_{\text{line}}I_{\text{line}}\left(-\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta\right)$$

$$= \sqrt{3}\, E_{\text{line}}I_{\text{line}} \sin \theta_{\text{coil}}. \qquad (18)$$

Thus the meter with the phasing transformer records Eq. (18) of Art. 39 directly without the use of a multiplying factor.

41. Measuring Reactive Kilovolt-amperes in a Three-phase, Four-wire System.

As previously stated, the usual purpose of measuring reactive kilovolt-ampere is to obtain the power factor. The complete meter installation for this purpose requires two meters. One records kilowatt-hours and the other reactive kilovolt-ampere-hours. The two meters are then combined in Eq. (19), Art. 39, to obtain what is termed the average power factor. A metering scheme such as this is illustrated in Fig. 51. The system is three-phase, four-wire and standard two-element meters are used. Three current transformers connected in delta are used as in Fig. 47. The currents obtained by these transformers are led through the current coils of both meters in series. It will be remembered that the current I_{cC} circulates

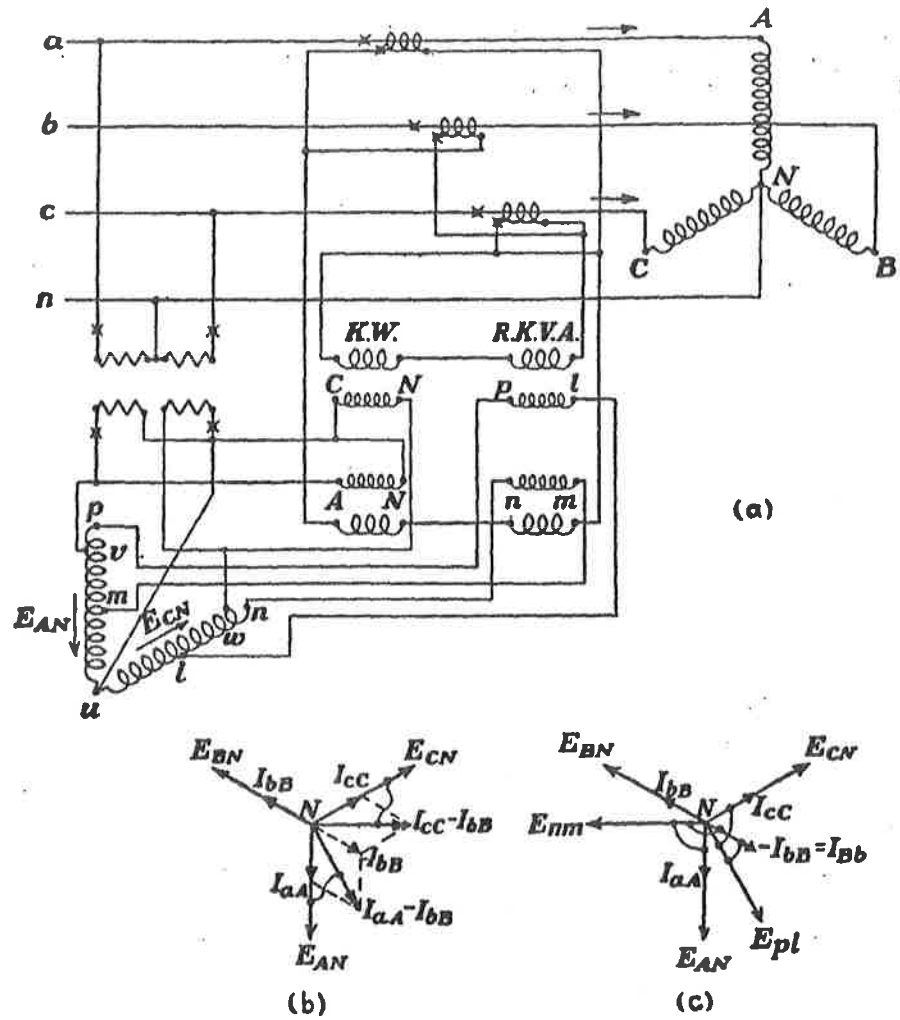

FIG. 51.—Reactive kilovolt-ampere measurement in a three-phase, four-wire system.

through the top current coils and I_{aA} through the bottom current coils. I_{bB} circulates backward through all four coils. Voltage transformers are also shown. They are connected line to neutral on the high side and open delta on the low side. The kilowatt-hour meter is connected through the voltage transformer as in Fig. 47*a*. Figure 51*b* shows again the vector diagram for this connection with currents of unity power factor. A phasing transformer similar to that employed in Fig. 50 is used to produce the necessary quadrature voltages for the reactive kilovolt-ampere meter. In this case the voltages from line to neutral are applied between the points *vu* and *wu*. As before, the voltages *nm* and *pl* are equal to the voltages applied to *vu* and *wu* but at right angles to them. In this case they are equal to coil voltages, not line voltages. The potential coils of the reactive kilovolt-ampere-hour meter are so connected to this transformer that the top element has applied to it voltage E_{pl} equal to and at right angles to E_{cn}. The bottom element has applied to it the voltage E_{nm} equal to and at right angles to E_{AN}. This is illustrated in Fig. 51*c* with currents at unity power factor.

Assuming balanced symmetrical voltages and unbalanced currents, we have the top and bottom elements recording, respectively,

$$Q_T = E_{pl}I_{cC} \cos (90 - \theta_C) + E_{pl}I_{Bb} \cos (30 - \theta_B) \text{ and } Q_B = E_{nm}I_{aA} \cos (90 - \theta_A) + E_{nm}I_{Bb} \cos (150 - \theta_B).$$

The meter as a whole reads

$$Q = E_{coil}I_{cC} (\cos 90 \cos \theta_C + \sin 90 \sin \theta_C) + E_{coil}I_{aA} (\cos 90 \cos \theta_A + \sin 90 \sin \theta_A) + E_{coil}I_{Bb} (\cos 30 \cos \theta_B + \sin 30 \sin \theta_B + \cos 150 \cos \theta_B + \sin 150 \sin \theta_B)$$

$$= E_{coil} I_{cC} (0 \cos \theta_C + 1 \times \sin \theta_C) + E_{coil} I_{aA} (0 \times \cos \theta_A + 1 \times \sin \theta_A) + E_{coil} I_{Bb} \left(\frac{\sqrt{3}}{2} \cos \theta_B + \frac{1}{2} \sin \theta_B - \frac{\sqrt{3}}{2} \cos \theta_B + \frac{1}{2} \sin \theta_B \right)$$

$$= E_{coil} I_{cC} \sin \theta_C + E_{coil} I_{aA} \sin \theta_A + E_{coil} I_{Bb} \sin \theta_B.$$

This equation is similar to Eq. (16), Art. 38, except that the sine takes the place of the cosine. Thus, instead of power, it records reactive volt-amperes. As in Eq. (16), as long as the voltages remain balanced and equally spaced, the currents and their angles may take on any value.

PART IV

POWER-FACTOR CORRECTION

CHAPTER IX

GENERAL PHASES OF THE POWER-FACTOR CORRECTION PROBLEM

The study of power-factor correction is both a technical and an economic problem. It is closely allied to the meter department and hence the meter engineer. The growing appreciation of the benefits of high power factor by the power-consuming public and the light and power industry alike has led to a more extensive use of power-factor clauses in rate schedules. As soon as a customer decides that there is an economic advantage to him in installing power-factor corrective apparatus, there immediately arises a metering problem which the meter engineer and meter department must handle. It follows, therefore, that in order to have a real appreciation of his specific problem, the meterman should be conversant at least with the broader problem of power-factor correction methods and their advantages and disadvantages.

42. The Disadvantages of Low Power Factor.

The disadvantages of low power factor are many and varied. The three principal ones are: (1) increased kilovolt-ampere capacity of all current-carrying apparatus for a given kilowatt output. Such apparatus includes generating equipment, transformers, switchgear, transmission and distribution lines, and other associated apparatus; (2) the increased losses in such equipment; (3) poor voltage regulation over the entire system. To these might be added certain objectionable features in the design of apparatus required to be operated at low power factor. To illustrate, most electrical apparatus is limited in the load it can carry by its current-carrying capacity. As a result, the power (kilowatt) load which a given piece of apparatus can carry is proportional to the power factor. Thus a piece of equipment operating at 0.8 power factor can carry only 80 per cent of the kilowatt load that it could carry at unity power factor. By voltage regulation is meant the difference in voltage available between full load and no load. For instance, the regulation of a transformer is approximately 1 per cent at unity power factor and 3 per cent at 0.7 power factor. Another way of illustrating these two principal effects of low power is as follows: The line drop IR increases directly with the current, but the line loss I^2R increases as the square of the current. At 0.7 power factor the current is $I/0.7 = 1.43\ I = 43$ per cent greater than at unity power factor, and therefore IR is 43 per cent greater. The kilowatt loss at 0.7 power factor is $(1.43I)^2R = 2.04(I^2R)$, or approximately double the unity power-factor loss. It follows, therefore, that for the same line loss, the copper conductor will have to be twice the cross-section if operated at 0.7 power factor than if operated at unity power factor.

For the purpose of illustrating the benefits of power-factor correction, it is more striking to give the converse of the above statement. If our system is operating satisfactorily at an average power factor of 0.7, we can increase our kilowatt output by 43 per cent if we can increase our power factor to unity without increasing the losses in any of the associated apparatus.

43. Causes of Low Power Factor.

It will be well at this point to list the most common sources of low power factor. Fundamentally, low power factor is caused by induction apparatus. Such apparatus requires exciting or magnetizing current. This current is commonly called wattless current in contradistinction to energy-consuming current. The chief cause of low power factor is underloaded induction motors in industrial establishments. Electric furnaces, welders, transformers, and even transmission lines are other sources of low power factor. It is sometimes possible greatly to improve conditions by relocating machines or reorganizing production methods, but more often many machines, due to the nature of the work, cannot be operated at full load all of the time. They must be operated at greatly reduced load, resulting in inherent low power-factor conditions.

44. Methods of Correcting Low Power Factor.

At present the means available for correcting power factor are:

1. The use of standard-type induction motors properly loaded to give the highest possible power factor for the operating conditions of the application involved.

2. The use of synchronous motors either in addition to present equipment or, where possible, in place of such equipment.

3. The use of capacitors (static condensers).

4. The use of synchronous condensers (overexcited synchronous motors running idle).

45. Location of Power-factor Corrective Apparatus.

The place where power-factor corrective apparatus is installed depends upon local conditions and the economical gain to be obtained. A fundamental principle of such application is that power-factor corrective apparatus always improves the conditions back along the system toward the source of power—never forward from the corrective apparatus toward the consumption of the power. Thus power-factor correction directly at a source of the low power factor (such as an electric furnace) will improve to some extent the power factor of the main-line equipment from this point back to the generators at the power house. However, the power factor of the electric furnace itself is not affected. If such equipment is installed at the customer's transformer bank, it will improve the power factor of the distribution, transmission, and generating equipment back toward the generating plant but will not improve the power factor of the customer's local distribution system. This does not mean that the customer will not be benefited, as improved voltage regulation will nevertheless result. In like manner, such equipment installed at a substation will improve the power factor of the feeder and transmission system back toward the generating equipment but will not improve the power factor on the load side of the substation. Here again, however, improved voltage regulation will result on the load side.

46. Advantages and Disadvantages of the Various Types of Power-factor Correction Apparatus.

Which type of corrective apparatus to install in a given case depends not only upon the technical and economical aspects of the local situation but also upon the inherent advantages and disadvantages of the methods listed above. These will now be considered.

Although induction motors inherently draw lagging currents, they may nevertheless be used to improve power factor if properly installed and run as near full load as possible. They are exceedingly rugged in construction with consequent low maintenance cost and they possess suitable characteristics for most applications. These facts, coupled with their exceptionally low cost, make them the first choice when and where they will suffice. Synchronous motors are more expensive, less rugged in construction, and require greater maintenance cost. The need of direct current for the field is sometimes a disadvantage. Although their characteristics are not quite so universally applicable, they are becoming more generally used. They have greater corrective power than do induction motors with ease of control through the field rheostat. They can carry mechanical load as well as correct power factor, and at reduced load added corrective powers are available. Their constant-speed (no-slip) characteristic and high efficiency are advantages in some cases. Capacitors, or static condensers, require no attention whatever. They are very efficient (losses are less than one-half of 1 per cent). They may be applied directly at the source of low power factor with extreme flexibility as to size and location. They have the disadvantage of higher first cost due to lack of capacitor diversity factor, which results in higher total capacity for the application. Also there is often objection to having additional apparatus in the immediate vicinity of the motor or machine which it drives. Synchronous condensers are likely to be advantageous where the corrective condensive kilovolt-amperes can be applied at one point and in amounts exceeding 300 kilovolt-amperes. In this case they possess low first cost, inherent characteristics which tend to stabilize the voltage, and easy adjustment of the leading reactive kilovolt-amperes supplied. As compared to static condensers, they have greater losses and higher attendant and maintenance costs.

CHAPTER X

THE TECHNICAL CONSIDERATIONS OF POWER-FACTOR CORRECTION

Having discussed the more general aspects of the power-factor correction problem, we have now to consider the more specific technical phases of the subject.

47. Definition of Single-phase Power Factor.

A good deal has been said about power factor throughout this test and it will be well for the reader to review the material before proceeding further.[1] As previously stated, the fundamental cause of lagging power factor in an alternating-current circuit is the magnetizing current required to set up the magnetic fields associated with such a circuit. The proof, that the magnetizing current lags behind the voltage, requires the use of the calculus.[2] This is a branch of mathematics beyond the scope of this text. A similar statement holds for the proof that the current may sometimes lead the voltage. However, this lagging and leading phenomenon may be readily shown experimentally by the use of the oscillograph. The oscillograph is an instrument capable of producing on a photographic plate the actual instantaneous magni-

[1] See Chap. I, Art. 8; Chap. II, Arts. 14 and 17; Chap. V, Art. 29 and 30 and Chap. VIII, Art. 39.

[2] See Blalock, G. C., "Principles of Electrical Engineering," Chap. XVI, Art. 164.

tude of the voltage and current in a circuit. Thus an oscillogram of a circuit containing inductance will show photographically the actual relation between the voltage and current in that circuit. Such an oscillogram is reproduced in Fig. 52*a*. It will be noticed that the current lags behind the voltage by the time T. That is, the current does not start to build up in the

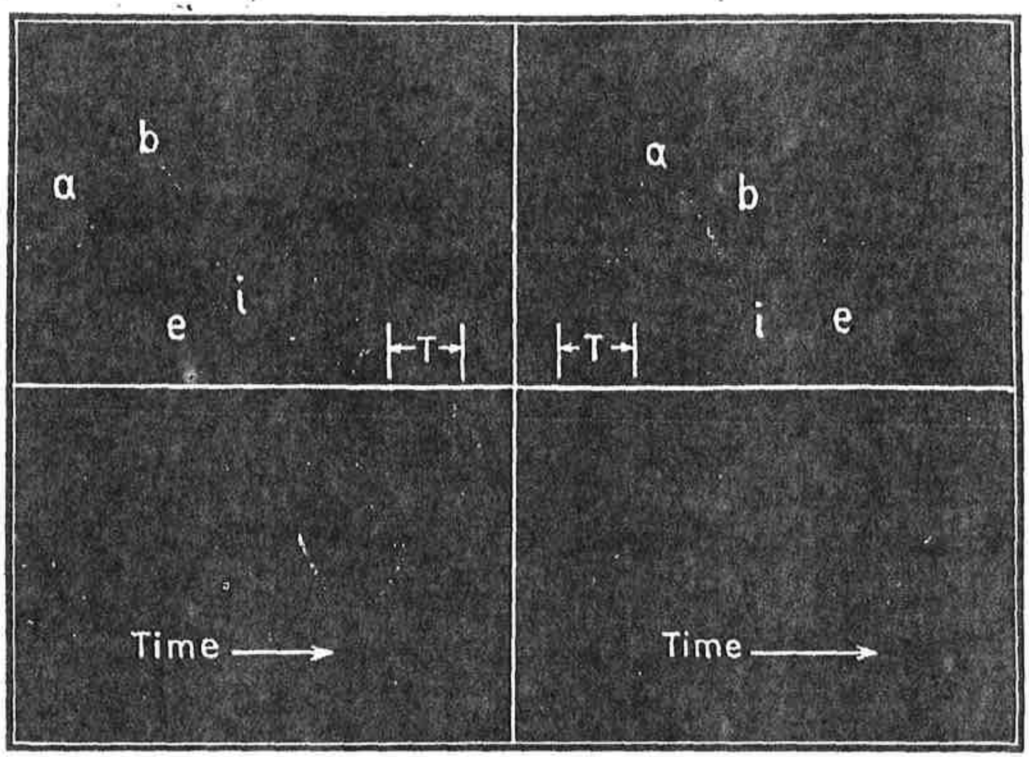

Fig. 52.—Oscillograms of the voltage and current in a circuit containing (*a*) inductance (*b*) capacity.

positive direction until T seconds after the voltage has started to build up in the positive direction. Figure 52*b* shows a similar oscillogram of a circuit-containing capacity. It will be noticed that here the current leads the voltage by the time T. In other words, the voltage wave e (Fig. 52*a*) reaches its positive maximum a before the current i reaches its positive maximum b. Thus the voltage leads the current or the current lags behind the voltage in time. Also, in Fig. 52*b*, the current *i* reaches its positive maximum *a* before the voltage *e* reaches its positive maximum *b*. Thus in this case the current leads the voltage in time.

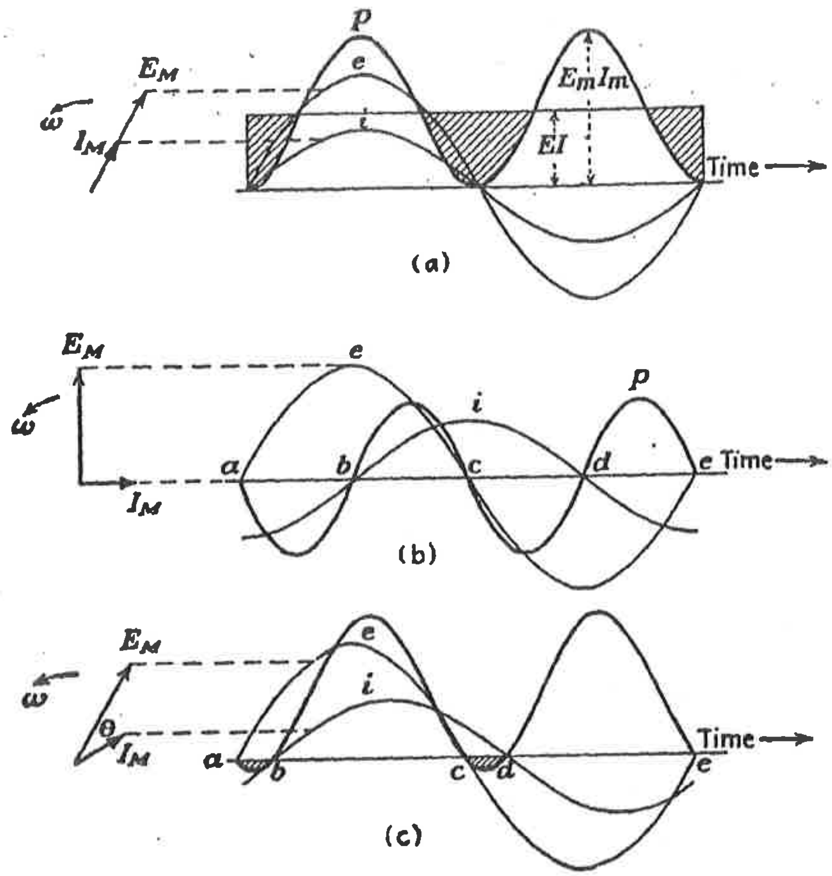

FIG. 53.—Instantaneous power curves.

In addition to the method previously used in the discussion of power factor, this effect can also be shown by the sign wave itself. In Fig. 53*a*, *e* is a sine wave of voltage and *i* a sine wave of current in phase with each other. The power at any instant of time is the product of the voltage and current at that same instant of time. Therefore if the voltage and current are multiplied together point for point along the time axis and the result plotted (in this case to a different scale), a power curve will result. Such a curve is curve p in Fig. 53*a*. This curve is all positive even when the current and voltage are negative, because the product of two negative quantities is always positive. It should be noticed that the power curve is also a sine wave but that it is double the frequency of the voltage or current wave. Since the voltage and current are in phase and therefore reach their maximum values at the same instant, the curve p will reach its maximum value of $E_m I_m$ at that same instant. Also, its horizontal axis of symmetry is displaced above the zero axis by a distance equal to EI, where E and I are the effective values of the voltage and current, respectively. Since this axis is the axis of symmetry of the power curve, the top halves of this curve are just equal to the shaded portions below the axis. Consequently, EI must be the average value of the power in the circuit.

Now consider Fig. 53*b*. In this diagram the current and voltage are 90 degrees out of phase with each other (current lagging). The power curve is determined exactly as in Fig. 53*a*. At points *a*, *b*, *c*, *d*, and *e* either the voltage or the current is zero and the power is consequently zero. Between *a* and *b* the voltage is positive but the current is negative, which makes the power negative; between *b* and *c* both the voltage and current are positive, so that the power is positive. Between *c* and *d* the voltage is negative and the current positive, and as their product is negative, the power will be negative. Between *d* and *e* both the voltage and current are negative, and since the product of two negative quantities is a positive quantity, the power is positive. Like the power curve of Fig. 53*a*, this curve is a sine wave of double frequency, but unlike the power curve of Fig. 53*a* its axis of symmetry coincides with the axis of symmetry of the voltage and current waves. The average power, therefore, is zero. That is, there is just as much negative power returned by the circuit as there is positive power given to the circuit. The average value of the power or the net value of the power delivered to the circuit is therefore zero.

Finally, in Fig. 53*c* are shown the same three curves as above. In this case, the current lags behind the voltage by an angle greater than zero but less than 90 degrees, namely 30 degrees. The effect is as might be expected—a compromise between the extreme values represented in Figs. 53*a* and *b*. There is some negative power but not so much as in Fig. 53*b*. There is considerable positive power but not so much as in Fig. 53*a*. It follows, then, that the average power is less than in Fig. 53*a* but more than in Fig. 53*b*. Since the only variable in these diagrams was the angle of lag between the voltage and current, it is obvious that the average power delivered to the circuit depends upon this angle. As previously stated, for single-phase circuits the power is proportional to the cosine of the angle of phase displacement between the voltage and current.

Thus:

$$P = EI \cos \theta$$

which is Eq. (5) of Art. 15, Chap II. In Fig. 53*a*,

$$\theta = 0$$

so that

$$\cos \theta = 1$$

and

$$P = EI \cos \theta = EI \times 1 = EI.$$

In Fig. 52*b*,

$$\theta = 90°,$$

so that

$$\cos \theta = 0$$

and

$$P = EI \times 0 = 0.$$

In Fig. 53*c*, however,

$$\theta = 30°,$$
$$\cos 30° = 0.866,$$

and

$$P = 0.866EI.$$

That factor by which the apparent power EI must be multiplied to produce the true power P is called the power factor. In the single-phase circuit it is also equal to the cosine of the angle of phase displacement between the voltage and current.

Thus:

$$\text{Power factor} = \cos \theta = \frac{P}{EI}.$$ [1]

48. Definition of Three-phase Power Factor.

If the three-phase system is balanced, the above definition still holds except for the factor $\sqrt{3}$.

Thus:

$$\text{Power factor} = \cos \theta = \frac{P}{\sqrt{3}EI}.$$ [2]

[1] See Chap. II, Art. 14.

[2] See Chap. V, Arts. 29 and 30.

If the three-phase system is unbalanced, it is necessary to proceed by the method of Art. 39, Chap. VIII, to a more general definition of power factor.[1] This process is again illustrated in Fig. 54. The diagrams in the first column of Fig. 54 are the vector diagrams of a single-phase circuit with lagging and leading currents. The second column of diagrams is the corresponding power triangles for these single-phase circuits; and the third column of diagrams is a set

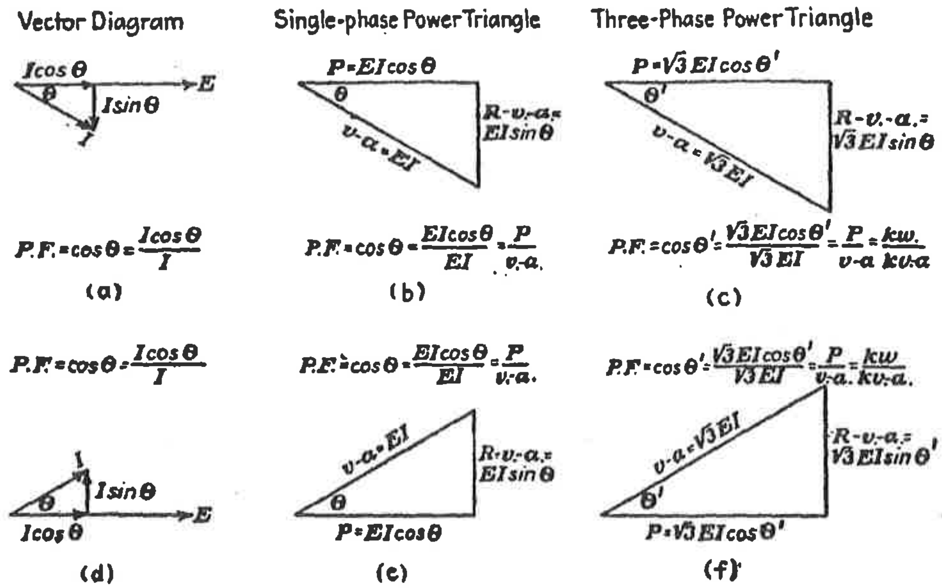

FIG. 54.—The power triangle.

of similar power triangles for a three-phase circuit. These diagrams will now be explained. Diagram *a* shows a single-phase circuit drawing a lagging current I which may be resolved into two components: $I \cos \theta$ in phase with the voltage, and $I \sin \theta$ in quadrature (at right angles) with the voltage. The power factor as defined in the previous article is equal to the cosine of theta, which, in turn, is equal to the in-phase component of the current divided by

[1] See Eq. (20), Chap. VIII, Art. 39.

the current. If the sides of this little current triangle are now multiplied by the voltage, a larger but similar triangle will be formed (diagram *b*). The hypotenuse of this triangle is equal to EI, called the volt-amperes, or the apparent power; the base is equal to $EI \cos \theta$, or the power; and the altitude is equal to $EI \sin \theta$ or the reactive volt-amperes. Since the only change between the current triangle of diagram *a* and the power triangle of diagram *b* is in the magnitude of the sides of the triangle which have all been increased by the same amount, the angle θ of diagram *b* equals the angle θ of diagram *a*. The power factor is then still equal to $\cos \theta$, which, in turn, is equal to $EI \cos \theta$ divided by EI. Since $EI \cos \theta$ is the power delivered to the circuit and EI is the volt-amperes of the circuit, it follows that the power factor is also equal to $\frac{P}{\text{volt-amperes}}$. As stated above, for a balanced three-phase circuit, the only change in the power-triangle diagram *c* over diagram *b* is a uniform increase in the sides of the triangle by the introduction of the factor $\sqrt{3}$.

Therefore:

$$\theta' = \theta$$

and

$$\text{Power factor} = \cos \theta' = \frac{EI \cos \theta'}{EI} = \frac{P}{\text{volt-amperes}},$$

as before, but if the system is unbalanced, then the phase power factors may all be different. Thus θ may have one value for phase *A*, a different value for phase *B*, and still a third value for phase *C*. The result is that θ' is not necessarily equal to any of them but is an angle formed by the triangular relationship of the volt-amperes, power, and reactive volt-amperes of the circuit. It should be carefully noted, however, that even though θ' has no longer any particular significance, the power factor of such a circuit may be defined by analogy as the ratio $\frac{P}{\text{volt-amperes}}$. Since we may shorten the sides of a triangle uniformly as well as lengthen them without changing the magnitude of the angles included therein, it follows that we may divide each side of the triangle in diagram *c* by 1,000 and define our power factor as $\frac{\text{kilowatts}}{\text{kilovolt-amperes}}$. This is the most general definition of power factor that it is possible to give and holds equally well for single-phase or three-phase balanced or unbalanced circuits. Since the hypotenuse of a right-angle triangle is equal to the square root of the sum of the squares of the sides, the following relations hold:

$$\text{Kv.-a.} = \sqrt{\text{kv.}^2 + \text{r kv.-a.}^2} \qquad (19)$$

and

$$\text{Power factor} = \frac{\text{kw.}}{\text{kv.-a.}} \qquad (20)$$

These are the Eqs. (19) and (20) of Art. 39, Chap. VIII. Diagrams *d*, *e*, and *f* show the same relationships as diagrams *a*, *b*, and *c* for a leading current. A comparison of these figures will show that the same definitions and equations hold.

49. The Principle of Power-factor Correction.

The principle of power-factor correction is the neutralizing of the lagging reactive component of the current either in whole or in part by a leading reactive component of current. Whether we choose to correct the power factor completely by introducing into the circuit a leading reactive component of current equal to the lagging reactive component already there or to correct the power factor only partially by introducing a small leading reactive component of current depends upon local conditions and the economic conditions of the problem (see Chap. XI). Which-

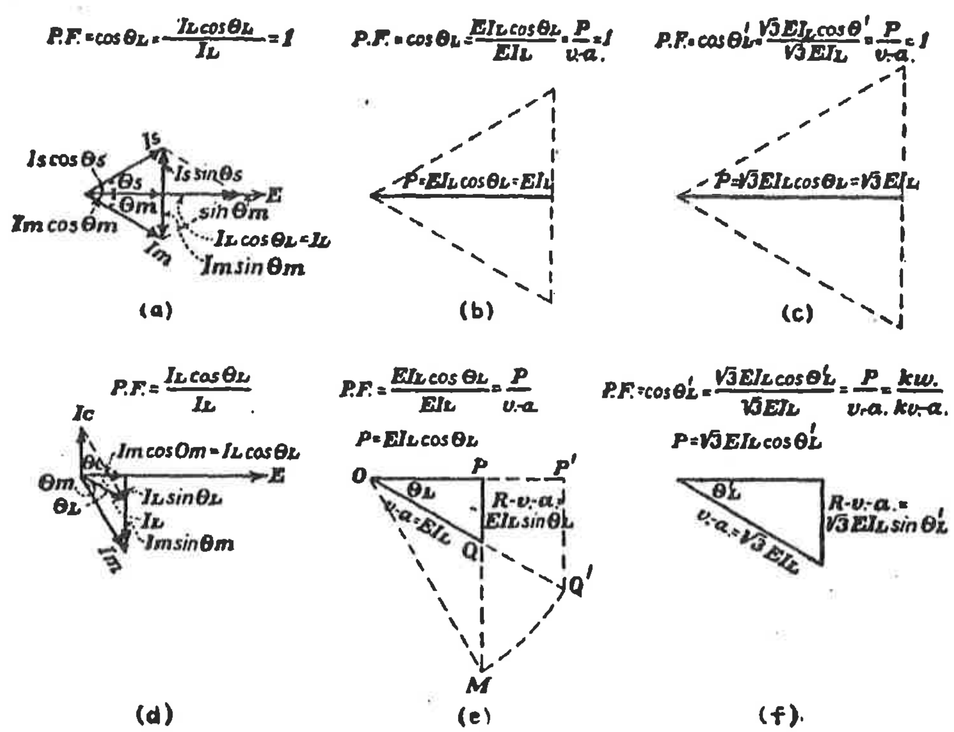

FIG. 55.—Power-factor correction diagrams.

ever we do, the principle of correction is the same; only the extent of correction differs. This principle will now be illustrated and explained.

In Fig. 55, two cases are considered. The first is a case of complete correction of power factor by a synchronous motor partly loaded; and the second case, partial correction by a static condenser. Diagram *a* shows an induction motor current I_m lagging behind

Thus correcting the power factor not only improves or decreases the angle of lag of the current behind the voltage but also makes available additional kilovolt-ampere capacity. In addition, the minor advantage of improved voltage regulation will result.

50. The Practical or Working Power-factor Correction Diagram.

In Art. 49 the fundamental principle of power-factor correction was illustrated by the vector diagram and the corresponding power triangles. In the practical solution of such problems, the vector diagram is frequently omitted. The power triangles alone are used in various combinations, as will now be illustrated. For the sake of custom, these diagrams are often turned upside down. Thus a power triangle representing an inductive or lagging power-factor load will be drawn upward above the horizontal axis, and a power triangle representing a leading power-factor load will be drawn downward. This is done primarily because we are accustomed to having lagging quantities drawn behind in a counterclockwise rotation diagram. Thus in a circuit where the current is lagging behind the voltage, we draw the kilowatts of that circuit lagging behind the kilovolt-amperes of the circuit just as we would draw the current lagging behind the voltage. Vectorially, this is not correct, but it is more natural and makes no difference so long as the kilowatt line is kept horizontal.

To illustrate such a diagram, assume that a certain consumer has an average load of 800 kilowatts at 60 per cent lagging power factor and that it becomes necessary to install an additional 500-horse power motor to provide rated additional mechanical power. The operating conditions are such that this customer has the choice of the following three possibilities.

a. A 500-horse power induction motor having an efficiency of 92 per cent and a power factor of 80 per cent lagging.

b. Installation of the same motor with a static condenser to raise the combined power factor of motor and condenser to 95 per cent lagging.

c. A 500-horse power synchronous motor having a full-load efficiency of 90 per cent, which operates at unity power factor.

Since in this article we are considering only the technical phases of power-factor correction, we shall compare the three-possibilities listed above as to the amount of power-factor correction possible for the customer at his transformer bank and leave the economical aspects of the problem for subsequent consideration.

The first step is to determine the power triangle of the original load. The kilovolt-amperes of the original load is

$$\frac{\text{Kw.}}{\text{P.f.}} = \frac{800}{0.6} = 1{,}333.$$

The power factor is

$$\cos \theta = 0.6, \ \theta = 53°$$

and

$$\sin \theta = 0.8.$$

The lagging reactive volt-amperes is

$$\text{Kv.-a.} \times \sin \theta = 1{,}333 \times 0.8 = 1{,}067.$$

With a convenient scale and these figures, we construct triangle OAB of Fig. 56. This is the power triangle of the orignal load. For the reasons just discussed, this triangle is drawn upward but nevertheless represents a lagging power-factor load. The second step is to determine the power triangle of each of the three motor possibilities and add each separately to the

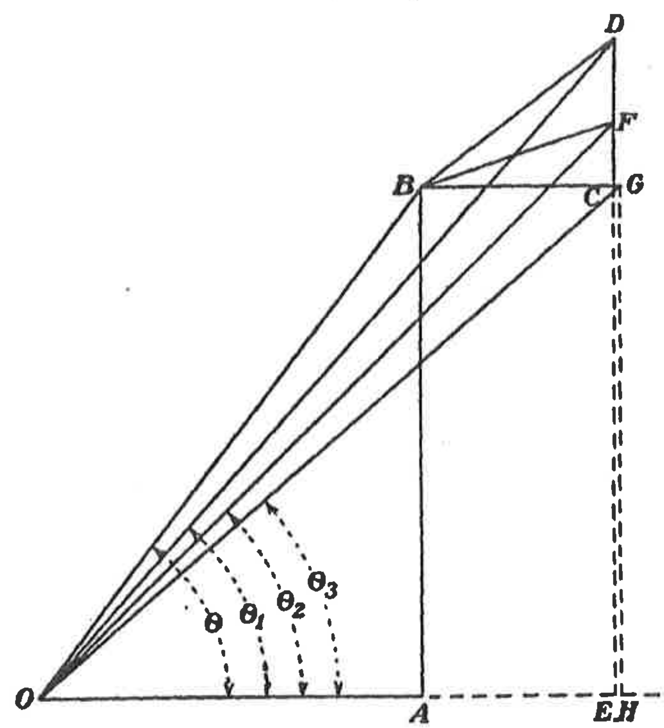

Fig. 56.—The practical or working power-factor correction diagram.

original power triangle OAB. For case a the kilowatt input to the induction motor is

$$\frac{\text{H.p.} \times 746}{\text{Eff.} \times 1{,}000} = \frac{500 \times 746}{0.92 \times 1{,}000} = 405.$$

The kilovolt-amperes are

$$\frac{\text{Kw.}}{\text{P.f.}} = \frac{405}{0.8} = 506.$$

Since the sine of the angle whose cosine is 0.8 is 0.6, the reactive kilovolt-amperes are

$$506 \times 0.6 = 304.$$

These figures, to the same scale, produce triangle *BCD*. Since the induction-motor load is in addition to the original load, the induction-motor power triangle is added on to the original triangle at its apex *B*. It follows that the new combined kilowatt load is

$$OE = OA + BC = 800 + 405 = 1{,}205.$$

The combined reactive kilovolt-amperes are

$$ED = AB + CD = 1{,}067 + 304 = 1{,}371.$$

The combined kilovolt-amperes are

$$OD = \sqrt{\text{kw.}^2 + \text{r kv.-a.}^2} = \sqrt{1{,}205^5 + 1{,}371^2} = 1{,}827.$$

The resultant power factor of the combination is

$$\cos \theta_1 = \frac{OE}{OD} = \frac{\text{kw.}}{\text{kv-a.}} = \frac{1{,}205}{1{,}827} = 0.65.$$

This is an improvement of 5 per cent over the original power factor.

For case *b* the kilowatt input to the motor will be the same as for case *a*, since the motor is doing the same work.

However, the kilovolt-amperes are

$$\frac{\text{Kw.}}{\text{P.f.}} = \frac{405}{0.95} = 426.$$

and the reactive kilovolt-amperes are

Kv.-a. $\times$ the sine of the angle whose cosine is 0.95

$$= 426 \times 0.31 = 132.$$

With these values and the same scale, triangle *BCF* is drawn. In this case the combined kilowatt load is still

$$OE = OA + AE = 800 + 405 = 1{,}205,$$

but the combined reactive kilovolt-amperes equal

$$EF = AB + CF = 1{,}067 + 132 = 1{,}199.$$

The combined kilovolt-amperes are

$$OF = \sqrt{\overline{OE}^2 + \overline{EF}^2} = \sqrt{\overline{1{,}205}^2 + \overline{1{,}199}^2} = 1{,}700.$$

The resultant power factor is

$$\cos \Theta_2 = \frac{OE}{OF} = \frac{1{,}205}{1{,}700} = 0.71.$$

This is an improvement of 6 per cent over the previous case and of 11 per cent over the original load. In case *c*, the kilowatt input to the synchronous motor is

$$\frac{\text{H.p.} \times 746}{1{,}000 \times \text{eff.}} = \frac{500 \times 746}{1{,}000 \times 0.9} = 414.$$

Since the kilovolt-amperes taken by the motor are equal to kilowatts divided by the power factor, it follows that when the power factor is unity, the kilovolt-amperes are equal to the kilowatts and the reactive volt-amperes are equal to zero. Thus the power triangle of the synchronous motor reduces to a straight line *BG*, 414 units long. The combined kilowatts are now

$$OH = OA + BG = 800 + 414 = 1{,}214,$$

the combined reactive volt-amperes are

$$HG = AB = 1{,}067,$$

and the combined kilovolt-amperes are

$$OG = \sqrt{\overline{OH}^2 + \overline{HG}^2} = \sqrt{\overline{1,214}^2 + \overline{1,067}^2} = 1,616.$$

The resultant power factor is

$$\cos \Theta_3 = \frac{OH}{OG} = \frac{1,214}{1,616} = 0.75.$$

This is an improvement of 15 per cent over the original load. Thus, as far as power-factor improvement is concerned, the synchronous motor is preferable.

51. The Synchronous Motor Operating at Power Factors Other than Unity.

In the previous article the synchronous motor was operated at unity power factor primarily for the

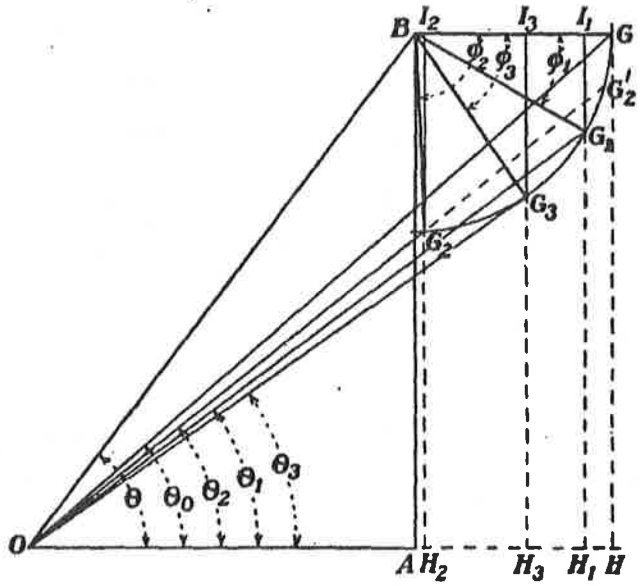

FIG. 57.—The synchronous motor operating as a synchronous condenser.

purpose of producing mechanical output. If the mechanical load is reduced on a synchronous motor and the field excitation increased in the right ratio, the machine may be made to draw a leading current of the same magnitude. The more the load is reduced and the field increased the more leading the current will be. If the load is reduced to zero and the field increased until the machine is drawing rated current, it will be found to be leading by nearly 90 degrees. When operating at this condition, the machine is called a synchronous condenser. The technical explanation of this phenomenon is too involved for consideration here, but the result, as far as power-factor correction is concerned, may be illustrated as in Fig. 57.

In this figure, triangle OAB represents the original load of the previous article and BG is the synchronous-motor input at rated load and unity power factor. As previously stated, the kilowatt input and kilovolt-amperes are equal at unity power factor. Hence, BG is also the rated kilovolt-amperes of the machine, and an arc with B as a center and BG as a radius will give the end position of the rated kilovolt-ampere line at whatever power factor the motor happens to be operating. Consider the motor operating at rated kilovolt-amperes but 0.87 power-factor leading. The angle of lead ϕ_1 of the motor kilovolt-ampere line BG_1 is then equal to the angle whose cosine is 0.87, or 30 degrees. When operating at this point, the leading reactive kilovolt-amperes are equal to G_1I_1 and the kilowatts drawn by the motor are equal to $BI_1 = AH_1$. It should be noticed that AH_1 is less than AH. In order not to overload the motor, some of the mechanical output must be sacrificed when it furnishes the leading reactive volt-amperes corresponding to G_1I_1. Now consider the motor operating at maximum leading power-factor angle ϕ_2. The only reason that ϕ_2 cannot be 90 degrees and therefore BG_2 vertically downward from B is that the motor must take some power to overcome its no-load losses. In Fig. 57 the no-load losses were assumed at 5 per cent, or

$$0.05 \times 414 = 21 \text{ kw.}$$

This power is required to operate the motor at no load. Therefore:

$$BI_2 = AH_2 = 21 \text{ kw.}$$

For this condition of operation the power input to the motor is reduced from AH to AH_2, but the leading reactive kilovolt-amperes have increased from zero at unity power factor to G_2I_2 at this point of maximum leading power factor of the motor. It should be noticed that although the motor is operating at full electrical load its mechanical output is zero.

$$\text{Output} = \text{input} - \text{losses} = 21 - 21 = 0.$$

The two points G_1 and G_2 will suffice to illustrate the fact that by proper adjustment of the field and mechanical output of the motor, it may be caused to operate anywhere along the arc GG_2 between the points G and G_2 and that when so operating the kilowatt input to the motor is the projection of the arc end of the motor kilovolt-ampere line upon the horizontal; and the leading reactive kilovolt-amperes, the projection of this point upon the vertical.

Returning, now, to the effect upon the power factor of the combined loads, when operating the motor at various leading power factors we proceed as in Art. 50 and Fig. 56. A line drawn from O to G, G_1, G_2, etc., represents the resultant kilovolt-amperes of the combined loads; and the angles θ_0, θ_1, θ_2, etc., the power-factor angle of the combination. The cosine of this angle is the power factor of the combination. In Fig. 57, Θ_0 is the same angle as Θ_3 in Fig. 56, and its cosine is 0.75. To find $\cos \Theta_1$ we proceed as follows:

$$\cos \phi_1 = 0.87,$$

therefore:

$$AH_1 = BI_1 = BG_1 \cos \phi_1 = 414 \times 0.87 = 360.$$

The combined kilowatts are

$$OA + AH_1 = 800 + 360 = 1{,}163.$$

reactive kilovolt-amperes of motor are

$$G_1I_1 = BG_1 \sin \phi_1 = 414 \times 0.5 = 207.$$

The combined reactive kilovolt-amperes are

$$H_1G_1 = H_1I_1 - G_1I_1 = AB - G_1I_1 = 1{,}067 - 207 = 860.$$

The combined kilovolt-amperes are

$$\sqrt{\text{kw.}^2 + \text{r kv.-a.}^2} = \sqrt{1{,}163^2 + 860^2} = 1{,}446.$$

The power factor is

$$\cos \Theta_1 = \frac{\text{kw.}}{\text{kv.-a.}} = \frac{1{,}163}{1{,}446} = 0.8.$$

Cos Θ_2 is found in a similar manner. The combined kilowatts are

$$OA + AH_2 = 800 + 21 = 821.$$

$$G_2I_2 = \sqrt{BG_2^2 - BI_2^2} = \sqrt{414^2 - 21^2} = 413.$$

The combined reactive kilovolt-amperes are

$$AB - G_2I_2 = 1{,}067 - 413 = 654.$$

The combined kilovolt-amperes are

$$\sqrt{821^2 + 654^2} = 1{,}049.$$

$$\cos \Theta_2 = \frac{821}{1{,}049} = 0.78.$$

Cos θ_2 is less than cos θ_1, and therefore the power-factor improvement is greater than when the motor was operated at G_1. At first thought, this appears to be contrary to what was expected, but if we produce the line OG_2 until it cuts the arc again at G_2', we find, so far as the resultant power factor of the combination is concerned, that it makes no difference whether the motor is operated at G_2 or G_2'. However, G_2' is nearer to G than G_1. Although the resultant improvement is less than at G_1, the mechanical power output of the motor is greater. This consideration raises the question as to the point of operation of the motor for maximum resultant power-factor improvement. For the size of motor used in Fig. 57 the maximum power-factor correction possible is when the motor is operating at G_3. The point G_3 is found by drawing a line from O tangent to the arc GG_2. (A line is tangent to an arc when it touches the arc at one place only and does not pass through it if extended.) This point of tangency in Fig. 56 is G_3. The value of θ_3 and hence of cos θ_3 may be determined graphically from the figure. In this case,

$$\theta_3 = 35°$$

and

$$\cos \theta_3 = 0.82.$$

The motor is operating at a leading power factor equal to

$$\cos \phi_3 = 0.56.$$

The kilowatt input at this point is

$$AH_3 = 236,$$

and if we assume an efficiency of 90 per cent, the output horse power is

$$\frac{236 \times 1{,}000 \times 0.9}{746} = 285.$$

A study of Fig. 57 will show that it is impossible to have maximum power-factor correction and maximum horse-power output of the correcting motor simultaneously. Either one may be obtained alone but not both together. Various combinations of motor-output and power-factor correction may be obtained. For maximum correction, the motor must be loaded to the extent indicated. The point of most economical operation depends upon the conditions at the point of application and the particular rate schedule in use. These considerations will be discussed in the next chapter.

CHAPTER XI

ECONOMIC CONSIDERATIONS OF POWER-FACTOR CORRECTION

In Chap X the technical phases of the power-factor correction problem were discussed. In this chapter the economic aspects of the problem will be considered. *Will it pay in dollars and cents in the long run?* is the question that in the last analysis must be answered. It is entirely possible that the method which gives the greatest power-factor correction may not be the most economical in the long run.

52. The Annual Cost as a Basis of Comparison.

To compare two or more alternative propositions, it is essential that they be compared on the same basis. The basis usually employed in cases of this kind is the annual cost basis. Thus if one proposition costs more per annum than another, the second is, generally speaking, preferable. Economically, the second is always to be preferred, but there may be certain so-called intangible advantages in favor of the first which will overshadow its economic disadvantage.

For convenience in calculations, it is customary to divide the annual cost into two parts called, respectively, fixed charges and operating expense. The fixed charges are proportional to the capital invested and are usually expressed as a percentage of the capital, namely, first cost plus installation cost. The fixed charges are made up of (*a*) interest on capital, (*b*) depreciation, (*c*) insurance, and (*d*) taxes. The operating expense, on the other hand, is usually proportional to the work done by the project or the amount of service rendered. Such items as wages, fuel, light, power, and maintenance are more nearly proportional to the service the project renders or the amount it is used. Such items therefore are grouped under the operating expense and vary to some extent with the nature of the project. Thus:

Annual cost = fixed charges + operating expense.

I. Fixed charges.
   a. Interest on capital expended on project.
   b. Depreciation of equipment.
   c. Insurance on the property.
   d. Taxes on the property.

II. Operating expense (items depend upon nature of project).
   a. Wages.
   b. Fuel, power.
   c. Heat, light, water.
   d. Maintenance and supplies.

These principles will first be illustrated by the commonplace problem of the man who is considering building for himself a house. He wishes, of course, to determine whether it will cost more to build or pay rent. Assume that he finds he can build a house suitable to his needs for $5,000. Now it makes no difference whether or not he actually has the $5,000 in cash or whether he has borrowed all or part of it; the cost of the use of this money is chargeable to the annual cost of the house. If we assume that he has the cash, he could invest it conservatively at, say, 6 per cent per annum. If, on the other hand, he chooses to use it to build a house, he loses the 6 per cent. Therefore the 6 per cent is a proper charge against the cost of the house. On the other hand, if he borrows the money, he can hardly do so for less than 6 per cent per annum. In the latter case, the 6 per cent is an obvious charge against the cost of the house. However, the reader must appreciate that in either case the cost of the use of the money invested is a legitimate charge against the house. Thus he finds that the house will cost him 6 per cent of the amount invested for the use of the money for that purpose, whether he actually has the cash or borrows it.

He must next consider what it will cost him to maintain the property; in other words, what he must set aside each year for depreciation. It is obvious that the house will not last forever. It will need paint occasionally and other such repairs. If he assumes the life of the house at 20 years, he will have to put aside \$250 a year to replace the house at the end of 20 years. This assumes straight-line depreciation. Two hundred and fifty dollars is 5 per cent of the cost of the house. Thus he arrives at a rate of 5 per cent for depreciation. In order to protect himself and his investment, he finds that the insurance that he must carry on the house will amount to 1 per cent of the cost of the house. Finally, he discovers that the taxes on his property will amount to 3 per cent of the cost of the property. The total rate of carrying charges, or so-called fixed charges, is then $6 + 5 + 1 + 3 = 15$ per cent. At this rate, the fixed charges amount to $\$5{,}000 \times 0.15 = \$750$ per annum. To this must be added any other annual expense which cannot be expressed as a percentage of the original investment.

Such expenses are termed operating expenses. A little consideration will suffice to show that the cost of heat, light, gas, and water will be the same whether he rents or builds, and therefore in this case the operating expense does not enter the consideration. Consequently, the total annual cost of building the house is \$750. If we assume that he can rent a similar house for \$50 a month, the annual cost of renting will be \$50 × 12 = \$600. Thus, there is an economic advantage in renting of \$150 per annum. The next thing for our man to consider is whether or not the intangible advantages of owning his own house are worth to him \$150 a year. The so-called intangible advantages are advantages which cannot be readily assigned a money value. Some of these intangible advantages are such things as:

1. Pride of ownership.
2. Improved standing in the community.
3. Insurance against being forced to move.
4. Greater borrowing power.
5. Freedom to make such changes and alterations as he sees fit.

As a second illustration of economic selection on the basis of annual cost, consider the following problem:

A manufacturer wishes to purchase a motor of 25-horse-power rating. Which of the two following proposals would you advise and why?

a. Motor A costs \$450 and has an efficiency of 80 per cent.

b. Motor B costs \$500 and has an efficiency of 85 per cent.

In both cases it is estimated that the motor will operate at full rated load 8 hours per day and 300 days per year with an energy cost of 6 cents per kilowatt-hour. Fixed charges are 15 per cent of the investment.

It is convenient to tabulate the calculations as shown on page 151.

Here there is evidently a saving of \$3,424.50 − \$3,234.53 = \$189.97 in favor of motor B. Although the fixed charges of motor B are higher than those of motor A, the operating expense of motor B is less than that of motor A due to the higher efficiency of motor B. The result is that motor B will pay the cost of the extra investment and in addition yield

$\frac{189.97}{50} = 3.8 = 380$ per cent on the additional investment required.

The details of applying the principle of annual cost to power-factor correction problems depend upon the particular problem. A problem which contemplates power-factor correction with an additional mechanical-power requirement will contain items not present in a problem contemplating power-factor correction only. Bearing in mind the fundamental principles of the annual cost method of comparison as explained and illustrated in this article, it should not be difficult to understand the minor modification of it as applied to the power-factor correction problems to follow.

53. The Importance of the Power-factor Clause in the Rate Schedule.

Ordinarily the advantages of improved service and increased energy capacity of feeders and associated apparatus are of secondary importance to the customer

TABLE I.—ECONOMIC SELECTION OF A MOTOR

Item	Calculation	
	Motor A	Motor B
Fixed charges	$450 \times 0.15 =$ \$ 67.50	$500 \times 0.15 =$ \$ 75.00
Operating expense	$\frac{25 \times 746 \times 8 \times 300 \times 0.06}{0.80 \times 1,000} =$ 3,357.00	$\frac{25 \times 746 \times 8 \times 300 \times 0.06}{0.85 \times 1,000} =$ 3,159.53
Annual cost	\$3,424.50	\$3,234.53

who contemplates improving his power factor. The principal economic advantages to him are either the reduction in his present power bill or the installation of additional power without increasing his present power bill appreciably. Therefore, unless the rate schedule under which he operates has a power-factor clause liberal enough to make it economically possible for him to accomplish one or the other of the above advantages, he will not install power-factor corrective apparatus. Thus the power-factor clause is of utmost importance, since from this source must come the gross revenue out of which to pay the annual cost of the corrective apparatus and any net gain that may result.

Although power-factor clauses in rate schedules differ widely, the following is typical: An average power factor is determined, say 80 per cent, and if the customer raises his power factor above this value at the time of his maximum demand, his demand for billing purposes is reduced by the ratio of the average power factor to the actual power factor. On the other hand, if he falls below the average at the time of his maximum demand, his actual demand for billing purposes is increased by the ratio of the average to the actual power factor. Thus, suppose a customer has an actual demand of 2,000 kilowatts and his power factor at the time of this demand was 90 per cent. His billing demand would be $2{,}000 \times {}^{80}/_{90} = 1{,}780$ kilowatts. Or if his actual power factor happened to be 70 per cent at the time of this demand, his billing demand would be $2{,}000 \times {}^{80}/_{70} = 2{,}285$ kilowatts. The average power factor is usually determined over a relatively long period, say approximately a year.

54. Power-factor Correction with Additional Mechanical-power Requirement.

The following example will illustrate the analysis to be made to determine the correct choice of motor for a given load requirement where advantage can also be taken of power-factor correction.

The present average electrical power required by a consumer for a typical day of 8 hours during 300 days per year is 800 kilowatts at 60 per cent power factor and maximum demand of 1,000 kilowatts.

The rate schedule provides for a charge of $1.40 per kilowatt per month of billing demand. The billing demand is to be figured by multiplying the actual demand by the ratio of the average power factor of 70 per cent to the actual power factor expressed in per cent. In addition, energy costs 2 cents per kilowatt-hour per month.

It becomes necessary to install a 500-horse-power motor to provide rated additional mechanical power during all of the above operating time. It is estimated that the maximum demand will be increased by the kilowatt input to this motor and that the additional energy may be purchased at the same rate as the original load. Which of the following installations is preferable?

a. A 500-horse-power induction motor having an efficiency of 92 per cent and a power factor of 80 per cent lagging, at a cost of $10 per rated horse power installed. Fixed charges at 10 per cent.

b. Installation of the same motor with a static condenser to raise its power factor to 95 per cent at a total combined cost of $15 per horse power of motor rating. Fixed charges at 12 per cent.

c. A 500-horse-power synchronous motor to operate at full load. Such a motor has an average efficiency of 90 per cent, no load losses of 5 per cent, and is estimated to cost $12 per horse power installed. Fixed charges at 15 per cent.

The first step in the solution of this problem is to determine the resulting power factor for each case above. Since these three motors are the same three that were used in Chap. X and for which Figs. 56 and 57 were drawn, we can consider all the calculations of Chap. X as part of this problem and as having been done here. The second step is then to compare the annual cost of each case. On the assumption that either motor is equally satisfactory as far as local load requirements go, it may be assumed that the one with the lowest annual cost is preferable. It is convenient to arrange the calculations in tabular form as in Table II. Item 1 is the first cost of each case figured from the cost per horse power, as shown. Item 2 is the total fixed charges. Item 3 is the new power factor resulting from the combination of the new motor and original load as figured in Chap. X. Item 4, the new demand, is obtained by adding to the original demand the demand of the new motor. The billing demand (item 5) is obtained by multiplying the actual demand by the ratio of the average to the new power factor.

The demand charge (item 6) is obtained by multiplying the billing demand by the rate per kilowatt per month and then multiplying by the number of months in a year. Item 6 is the first item operating expense. Items 7 to 10 inclusive comprise the calculation of the energy charge, or second item in the operating expense. In this group of calculations, the average kilowatts is found by adding to the average of the original load, the average kilowatt consumption of the new machine. Item 11 is then the annual cost, consisting of the fixed charge (item 2) and the operating expense (items 6 and 10). Item 12 is, of course, the difference between the annual cost of the induction motor and the induction motor plus condenser. Item 13 is a similar calculation for the synchronous motor.

On the assumption that any of the three combinations will be satisfactory as far as the load requirements and local conditions are concerned, the synchronous motor shows a saving of $666.72 per year over its nearest competitor.

55. Power-factor Correction Using the Synchronous Condenser.

If the customer of Art. 54 did not require any additional mechanical power, would it pay him to install a 500-horse-power synchronous condenser? A synchronous motor of this size operating as a synchronous condenser would produce a resulting power-factor angle equal to θ_2 of Fig. 57. The motor itself would operate at point G_2 on the arc of Fig. 57 and at a leading power factor equal to cos ϕ_2. In this case the resultant power-factor cos θ_2 was found to be 78 per cent. In Table III below, a calculation is made similar to that of Table II. Here, however, the comparison is made between the annual cost of the original load and the annual cost of the

TABLE II.—ECONOMIC SELECTION OF BEST MOTOR COMBINATION WHEN ADDITIONAL POWER IS REQUIRED

Number	Item	Calculations: Induction motor	Calculations: Induction motor plus condenser	Calculations: Synchronous motor
1	First cost	500 × 10 = $ 5,000.00	500 × 15 = $ 7,500.00	500 × 12 = $ 6,000.00
2	Fixed charges	5,000 × 0.10 = $ 500.00	7,500 × 0.12 = $ 900.00	6,000 × 0.15 = $ 900.00
3	Corrected power factor	page 138 65 per cent	page 139 71 per cent	page 140 75 per cent
4	New demand	1,000 + 405 = 1,405	1,000 + 405 = 1,405	1,000 + 414 = 1,414
5	Billing demand	1,405 × 70/65 = 1,513	1,405 × 70/71 = 1,385	1,414 × 70/75 = 1,319.6
6	Demand charge	1,513 × 1.4 × 12 = $25,418.40	1,385 × 1.4 × 12 = $23,268.00	1,319.6 × 1.4 × 12 = $22,169.28
7	Average kilowatts	800 + 405 = 1,205	800 + 405 = 1,205	800 + 414 = 1,214
8	Hours per year	8 × 300 = 2,400	8 × 300 = 2,400	8 × 300 = 2,400
9	Kilowatt-hours	1,205 × 2,400 = 2,892,000	1,205 × 2,400 = 2,892,000	1,214 × 2,400 = 2,913,600
10	Energy charge	2,892,000 × 0.02 = $57,840.00	2,892,000 × 0.02 = $57,840.00	2,916,000 × 0.02 = $58,272.00
11	Total annual cost (item 2 plus 6 plus 10)	$83,758.40	$82,008.00	$81,341.28
12	Induction motor + condenser over induction motor	....	$ 1,750.40	
13	Synchronous motor over induction motor + condenser	....	....	$ 666.72

TABLE III.—ECONOMIC ADVANTAGE OF POWER-FACTOR CORRECTION WHEN ADDITIONAL POWER IS NOT REQUIRED. USE OF THE SYNCHRONOUS CONDENSER

Number	Item	Calculations: Original load	Calculations: Synchronous motor
1	First cost	....	500 × 12 = $ 6,000.00
2	Fixed charges	....	6,000 × 0.15 = 900.00
3	Corrected power factor	....	page 143 78
4	New demand	1,000	1,000 + 21 = 1,021
5	Billing demand	1,000 × 70/60 = 1,167	1,021 × 70/78 = 916
6	Demand charge	1,167 × 1.4 × 12 = $19,605.60	916 × 1.4 × 12 = $15,388.80
7	Average kilowatts	800	800 + 21 = 821
8	Hours per year	8 × 300 = 2,400	8 × 300 = 2,400
9	Kilowatt-hours	800 × 2,400 = 1,920,00	821 × 2,400 = 1,970,400
10	Energy charge	1,920,000 × 0.02 = $38,400.00	1,970,400 × 0.02 = $39,408.00
11	Total annual cost (item 2 plus 6 plus 10)	$58,005.60	$55,696.80
12	Advantage of power-factor correction	....	$ 2,308.80
13	Yield or per cent on investment	....	2,308.8/6,000.0 × 100 = 38.46 per cent

original load plus the synchronous condenser. It is found that an annual saving of \$2,308.80 results from such an installation. This saving amounts to 38.46 per cent on the investment required.

It must not be assumed that a yield of this magnitude is always possible. These problems are primarily for the purpose of illustrating the principles of economic selection and hold only for the particular set of conditions therein assumed. Such a set of conditions might never exist in practice. However, whatever the actual conditions might be they would enter into the problem exactly as they do in these examples, each different set of conditions producing a different result.

56. Power-factor Correction to a Given Power Factor with Additional Mechanical Power Requirement.

It sometimes happens that a customer wishes to do two things at once—provide for some additional mechanical power and correct the combined power factor to a specific value. What size synchronous motor to use and will it pay are the two logical questions to be answered in such a case.

Assume that the customer of Art. 54 wishes to provide 500 horse power additional mechanical power and also to raise the combined power factor of his original load and new motor to 80 per cent. What size motor will he need? This question involves a technical problem.

If the motor has an efficiency of 90 per cent, as before, the kilowatt input will be equal to

$$\frac{500 \times 746}{0.90 \times 1{,}000} = 414 \text{ kw.}$$

The combined kilowatts of original load and new motor will then be

$$800 + 414 = 1{,}214.$$

If the power factor of this combined load is to be 80 per cent, the kilovolt-amperes of the combined load are

$$\frac{\text{Kw.}}{\text{P.f.}} = \frac{1{,}214}{0.8} = 1{,}518.$$

The reactive kilovolt-amperes of the combined load is then the kilovolt-amperes times the sine of the angle whose cosine is 0.8, or

$$1{,}518 \times 0.6 = 911.$$

The reactive kilovolt-amperes of the original load was 1,067. If this is to be reduced to 911, the synchronous motor must furnish

$$1{,}067 - 911 = 156 \text{ reactive kilovolt-amperes.}$$

The kilovolt-amperes of the synchronous motor is then equal to

$$\sqrt{\text{kw.}^2 + \text{r kv.-a.}^2} = \sqrt{414^2 + 156^2} = 442.$$

The horse-power rating is

$$\text{P.f.} \times \frac{\text{kv.-a.} \times \text{eff.} \times 1{,}000}{746} = 1 \times \frac{442 \times 0.9 \times 1{,}000}{746} = 532.$$

(In practice it would not be advisable to attempt an odd-sized motor. We should go to 550 horse power.) The above quantities are shown graphically in the diagram of Fig. 58. Triangle *OAB* is the original load and is equal to triangle *OAB* of Figs. 56 and 57. *BI*, the kilowatt input to the synchronous motor, = 414. The reactive kilovolt-amperes of the synchronous motor as found above is

$$GI = 156.$$

GH is the reactive kilovolt-amperes of the resultant load, *OG* the resultant kilovolt amperes, and cos θ_1 the resultant power factor.

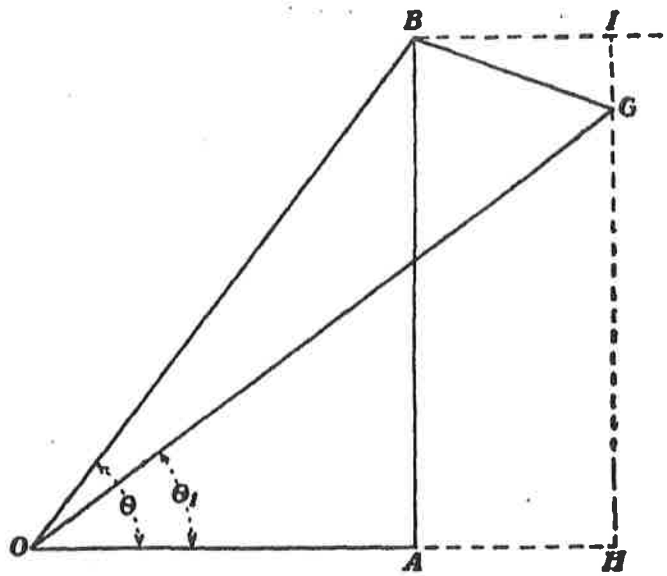

Fig. 58.—Use of a synchronous motor to deliver additional mechanical horse power and correct the power factor to 80 per cent.

To determine the cost of this installation, we proceed as in Table III. The result is tabulated in Table IV.

Comparing the synchronous motor of Table IV with the synchronous motor of Table II, it is evident that the additional investment of $600 to raise the combined power factor from 75 to 80 per cent will pay for itself two and one-half times in the first year. The indications are that it might pay this customer to install a synchronous motor large enough to give him the 500-horse power additional mechanical power and also correct the power factor to a still higher value.

TABLE IV.—POWER-FACTOR CORRECTION TO PROVIDE ADDITIONAL MECHANICAL POWER AND CORRECT TO A SPECIFIED POWER FACTOR

Number	Item	Calculation: synchronous motor
1	First cost	500 × 12 = $ 6,600.00
2	Fixed charges	6,600 × 0.15 = 990.00
3	Corrected power factor	80 per cent
4	New demand	1,000 + 414 = 1,414
5	Billing demand	$1{,}414 \times {}^{70}/_{80}$ = 1,237
6	Demand charge	1,237 × 1.4 × 12 = $ 20,581.60
7	Average kilowatt	800 + 414 = 1,214
8	Hours per year	8 × 300 = 2,400
9	Kilowatt-hours	1,214 × 2,400 = 2,913,600
10	Energy charge	2,913,600 × 0.02 = $ 58,272.00
11	Total annual cost	= $ 79,843.60
12	Advantage over synchronous motor of Table I	$ 1,497.68
13	Yield on additional investment	1,497/600 × 100 = 249.5 per cent

However, this cannot be determined for certain without working through the problem again for, say, a power factor of 90 and 100 per cent. The method employed would be similar to that outlined above for a power factor of 80 per cent.

57. Power-factor Correction to a Specified Value Using the Synchronous Condenser.

In the final example, let it be supposed that the customer of Art. 54 wishes to correct his power factor to unity and does not require additional mechanical power. This customer would naturally ask, what size synchronous condenser will be required and will it pay to install it?

In Fig. 59, triangle OAB is again the power triangle of the original load. To raise the power factor of the combined loads to unity, the synchronous condenser must reduce the angle θ to zero. To do this its kilovolt-ampere rating must be BG units long. AG is the no-load losses of the machine and IG the reactive kilovolt-amperes which the machine is furnishing. This reactive kilovolt-amperes must equal the reactive kilovolt-amperes of the original load. If it does not, the resultant kilovolt-amperes of the combination OG will not coincide with the line OA, and the angle θ will not be zero.

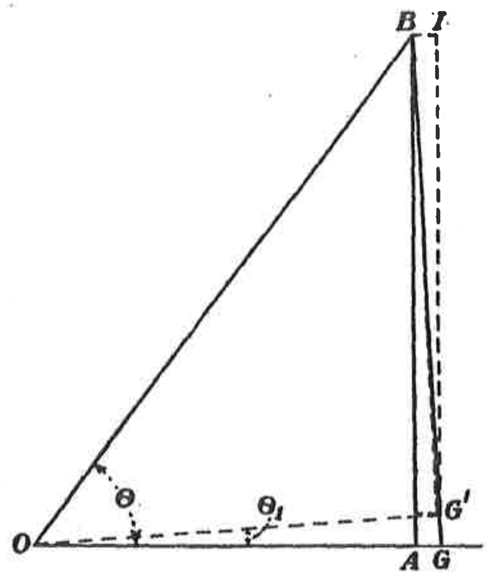

Fig. 59.—Use of the synchronous condenser to correct the power factor of a given load to unity.

Assuming 5 per cent no-load losses,

$$AG = 0.05\ BG,$$

and from Art. 54,

$$GI = 1{,}067.$$

Therefore:

$$BG = \sqrt{(0.05\ BG)^2 + (1{,}067)^2}$$

$$\overline{\text{kv.-a.}}^2 = \overline{BG}^2 = 0.0025\overline{BG}^2 + 1{,}138{,}489$$

$$0.9975\overline{BG}^2 = 1{,}138{,}489$$

$$\overline{BG}^2 = 1{,}141{,}442$$

$$\overline{BG} = 1{,}068$$

Theoretically, it would take a synchronous condenser of 1,068 kilovolt-ampere rating. In practice we should probably specify a 1,000 kilovolt-ampere machine. The horse-power rating of this machine at 90 per cent efficiency would be, $\frac{1{,}000 \times 1{,}000 \times 0.9}{746} \times 1.0 = 1{,}200$ horse power. If we reduce the machine from the theoretical requirement of 1,068 kilovolt-amperes to that of 1,000 kilovolt-amperes we shall not correct the power factor to unity. However, the difference is negligible, as is shown by the dotted lines in Fig. 59. G would move to G' and the resultant power factor would be less than unity by one minus the cosine of the angle θ'. The actual power-factor $\cos \theta'$ will be found to be 99.6 instead of 100 per cent. Therefore, the change is four-tenths of 1 per cent. For simplicity we shall assume that it is 100 per cent even with a 1,000 kilovolt-ampere machine. The no-load losses in this machine are $1{,}000 \times 0.05 = 50$ kilowatts.

To determine the economical value of such correction we prepare another table similar to Table III.

Table V shows only a moderate economic advantage in correcting to 100 per cent power factor over

TABLE V.—POWER-FACTOR CORRECTION TO A SPECIFIED VALUE USING A SYNCHRONOUS CONDENSER

Number	Item	Calculations: original load	Calculations: synchronous motor
1	First cost		1,200 × 12 = $14,400.00
2	Fixed charges		14,400 × 0.15 = 2,160.00
3	Corrected power factor		100
4	New demand	1,000	1,000 + 50 = 1,050
5	Billing demand	1,000 × $^{70}/_{60}$ = 1,167	1,050 × $^{70}/_{100}$ = 735
6	Demand charge	1,167 × 1.4 × 12 = $19,605.60	735 × 1.4 × 12 = $12,348.00
7	Average kilowatt	800	800 + 50 = 850
8	Hours per year	8 × 300 = 2,400	8 × 300 = 2,400
9	Kilowatt-hours	800 × 2,400 = 1,920,000	850 × 2,400 = 2,040,000
10	Energy charge	1,920,000 × 0.02 = $38,400.00	2,040,000 × 0.02 = $40,800.00
11	Total annual cost (items 2 plus 6 plus 10)	= $58,005.60	$55,308.00
12	Advantage of power factor correction to 100 per cent		$ 2,697.60
13	Advantage of power factor correction to 78 per cent Table II		$ 2,308.80
14	Additional advantage per year of correcting to 100 per cent		$ 388.80
15	Yield of 100 per cent correction over 78 per cent correction		$\frac{388.8}{14,400 - 6,000} \times 100$ = 4.6 per cent

correcting to 78 per cent power factor. In fact, when the intangibles are considered, such as space required for the larger machine and maintenance, it might not be advisable at all.

Most rate schedules are not so liberal as the one assumed here and it is seldom advisable or economically advantageous to correct power factor to the limit as in this last case. As stated previously, the reader should place the emphasis on the method of attack and solution of these problems and not on the numerical results. The results of such calculations, if not tempered with good engineering judgment, are practically useless.

APPENDIX

TRIGONOMETRIC FUNCTIONS

The trigonometric functions of an angle are the ratios to one another of the various sides of a right triangle having the given angle as one of its angles. There are, all told, six of these functions or ratios but only three of general application. The three most important trigonometric functions of the angle θ are defined as follows: In any right triangle containing the angle θ, such as triangle OAB of Fig. 1,

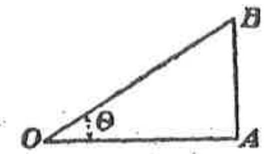

FIG. 1.

$$\text{sine of } \theta \text{, abbreviated } \sin\theta = \frac{AB}{OB},$$

$$\text{cosine of } \theta \text{, abbreviated } \cos\theta = \frac{OA}{OB},$$

$$\text{tangent of } \theta \text{, abbreviated } \tan\theta = \frac{AB}{OA}.$$

Since these functions are ratios, their value will be the same for any triangle, no matter how big, provided θ remains constant. Of course, they will have different values for different values of θ. By means of these ratios the value of an angle can be determined by the lengths of the sides of a triangle embracing the said angle. Conversely, the sides of the triangle OAB may be determined from the functions of one of its angles and one side.

Thus:

$$\text{side } OA = OB \cos \theta$$

and

$$\text{side } AB = OB \sin \theta = OA \tan \theta,$$

also

$$OB = \sqrt{\overline{OA}^2 + \overline{AB}^2}.$$

The above definitions hold for acute angles only. An acute angle is an angle between 0 and 90 degrees inclusive.

For obtuse angles (angles greater than 90 degrees), the above definitions still hold, but the functions may

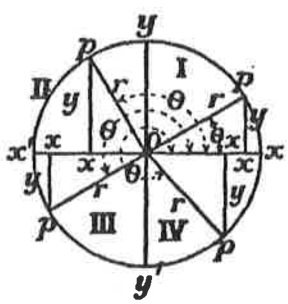

Fig. 2.

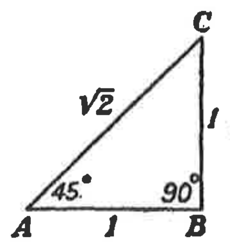

Fig. 3.

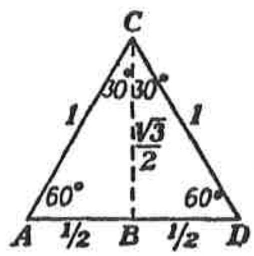

Fig. 4.

be positive or negative depending upon the location of the obtuse angle. In Fig. 2, the circle O of radius r is divided into four quarters, called quadrants, by a set of axes xx' and yy'. These quadrants are numbered I, II, III, and IV in the order shown. An angle is said to lie in the first, second, third, or fourth quadrant when its value falls between 0 and 90 degrees, 90 and 180 degrees, 180 and 270 degrees, and 270 and 360 degrees, respectively.

In this same figure we can represent an angle θ in each quadrant in turn by the angle (degrees) between the OX and the OP line as shown. In each case P

is r units from O, y units from the horizontal axis, and x units from the vertical axis.

The functions of the angle θ are then defined as follows:

$$\sin\theta = \frac{y}{r},$$

$$\cos\theta = \frac{x}{r},$$

$$\tan\theta = \frac{y}{x}.$$

Since quantities measured to the left of the vertical axis are negative, x will be negative in the above definitions for angles in quadrants II and III. Also, since quantities measured below the horizontal axis are negative, y will be negative in the above definitions for angles in quadrants III and IV. Thus the cosine of θ in quadrant II is

$$\frac{-x}{r} = -\cos\theta,$$

the sine of θ in quadrant III is

$$\frac{-y}{r} = -\sin\theta,$$

and the tangent of θ in quadrant IV is

$$\frac{-y}{x} = -\tan\theta, \text{etc.}$$

For convenience the sign of the various functions in the several quadrants as defined above are listed in Table I.

Another way of expressing these same relations is tabulated in Table II. Thus if θ is 120 degrees, it will fall in the second quadrant and is equal to 180 degrees − 60 degrees. Using Table II,

$$\cos 120° = \cos (180° - 60°) = -\cos 60°.$$

To find the sine of 240 degrees we note that 240 degrees falls in the third quadrant and is equal to

$$270° - 30°.$$

Table II gives

$$\sin 240° = \sin (270° - 30°) = -\cos 30°.$$

For angles greater than 360 degrees, the functions all repeat for the remaining angle after subtracting the nearest even multiple of 360 degrees.

Thus:

$$\sin 440° = \sin (440° - 360°) = \sin 80°$$

and

$$\sin 750° = \sin (750° - 2 \times 360°) = \sin 30°, \text{ etc.}$$

The above relations allow us to find the various trigonometric functions of any angle no matter how big. This can be done by using Table I with Fig. 2 or Table II.

Some of the limiting values of the functions may give trouble. For instance, if θ is zero, then, by Fig. 2,

$$\sin \theta = \frac{y}{r} = \frac{o}{r} = 0$$

and

$$\cos \theta = \frac{x}{r} = \frac{r}{r} = 1.$$

Now if

$$\theta = 90°,$$

$$\sin \theta = \frac{y}{r} = \frac{r}{r} = 1,$$

$$\cos \theta = \frac{x}{r} = \frac{o}{r} = 0,$$

and

$$\tan \theta = \frac{y}{x} = \frac{r}{o} = \text{a mathematical indeterminate which approaches infinity as a limit.}$$

The symbol for infinity is ∞. Thus in practice we say that

$$\tan 90° = \infty.$$

This, however, is a loose terminology. What we really mean is that as the angle θ approaches 90 degrees, the tangent increases without limit.

TABLE I.—ALGEBRAIC SIGNS OF THE FUNCTIONS

Location of angle	Sin	Cos	Tan
Quadrant I	+	+	+
Quadrant II	+	−	−
Quadrant III	−	−	+
Quadrant IV	−	+	−

TABLE II.—FUNCTIONS OF ANGLES IN ANY QUADRANT IN TERMS OF ANGLES IN THE FIRST QUADRANT

II

$\sin(180 - \theta) = \sin\theta$

$\cos(180 - \theta) = -\cos\theta$

$\tan(180 - \theta) = -\tan\theta$

$\sin(90 + \theta) = \cos\theta$

$\cos(90 + \theta) = -\sin\theta$

$\tan(90 + \theta) = \dfrac{-1}{\tan\theta}$

III

$\sin(270 - \theta) = -\cos\theta$

$\cos(270 - \theta) = -\sin\theta$

$\tan(270 - \theta) = \dfrac{1}{\tan\theta}$

$\sin(180 + \theta) = -\sin\theta$

$\cos(180 + \theta) = -\cos\theta$

$\tan(180 + \theta) = \tan\theta$

IV

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\sin(270 + \theta) = -\cos\theta$

$\cos(270 + \theta) = \sin\theta$

$\tan(270 + \theta) = \dfrac{-1}{\tan\theta}$

It is often convenient to have the values of the functions of some of the more common angles in whole figures rather than in decimals. Such values may readily be determined from the triangles of Figs. 2, 3, and 4. If in Fig. 3 the sides of the triangle *ABC* containing the 45-degree angle are 1, then the hypotenuse is $\sqrt{2}$,

$$\sin 45° = \frac{BC}{AC} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \frac{AB}{AC} = \frac{1}{\sqrt{2}},$$

and

$$\tan 45° = \frac{BC}{AB} = 1.$$

If in Fig. 4, triangle *ADC* is an equilateral triangle of sides equal to 1, a perpendicular dropped from *C* will bisect the base and be equal to $\sqrt{3}/2$.

Therefore:

$$\sin 60° = \frac{BC}{AC} = \frac{\sqrt{3}}{2},$$

$$\cos 60° = \frac{AB}{AC} = \frac{1}{2},$$

and

$$\tan 60° = \frac{BC}{AB} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Also from triangle *CBD* we get that

$$\sin 30° = \frac{BD}{CD} = \frac{1}{2},$$

$$\cos 30° = \frac{CB}{CD} = \frac{\sqrt{3}}{2},$$

and

$$\tan 30° = \frac{BD}{CB} = \frac{½}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

For angles greater than 90 degrees, Fig. 2 may be used, as explained above. The numerical values of the functions of some of the more common angles are tabulated in Table III.

TABLE III.—NUMERICAL VALUES OF THE FUNCTIONS OF SOME OF THE MORE COMMON ANGLES

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Sine....	0	½	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0	−1	0
Cosine .	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{3}/2$	−1	0	1
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞*	$-\sqrt{3}$	−1	$-1/\sqrt{3}$	0	∞	0

* The symbol ∞ stands for infinity.

There are a great many so-called trigonometric identities which give the interrelations of the various functions. They are convenient for the solutions of problems involving trigonometry. Of these, the following formulas are important for calculations in electricity. They give the sum and difference of two angles.

$$\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta.$$
$$\cos(\theta + \beta) = \cos\theta\cos\beta - \sin\theta\sin\beta.$$
$$\sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta.$$
$$\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\sin\beta.$$

β is the Greek letter Beta.

Example.—Find cos (θ + 30).

$$\cos(\theta + 30) = \cos\theta \cos 30 - \sin\theta \sin 30$$
$$= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta,$$

also

$$\cos(\theta - 30) = \cos\theta \cos 30 + \sin\theta \sin 30$$
$$= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta.$$

It would be extremely laborious to evaluate the trigonometric functions for each angle which we had occasion to use. Fortunately this is not necessary, as there have been prepared various tables of trigonometric functions which tabulate the numerical values of all angles from 0 to 90 degrees. For angles greater than 90 degrees the values may be obtained from the function of angles less than 90 degrees by any of the methods explained above. These tables sometimes list every degree and each minute of every degree, and the values are often worked out to the seventh decimal place. For our purpose, a three-place table listing each degree only will be sufficient. Such a table is given on page 176.

As examples of the use of this table consider the following:

1. Find the sine of 36 degrees. In the first column of the table we locate the angle 36 degrees. To the right and in the column whose heading contains the function we seek (in this case the second column) we find the value 0.588. This is the numerical value of the sine of 36 degrees.

2. The processes may be reversed. If we have the cosine of θ given as 0.2588, we may find the angle as follows. Under the column headed cosine, we look for the nearest number to 0.2588. In this way we find the value 0.259 in column seven; and to the left of this value, the angle 75 degrees. Therefore the angle whose cosine is 0.2588 is 75 degrees nearly.

Trigonometric Functions

A	Sin	Cos	Tan	*A*	Sin	Cos	Tan
0	0.000	1.000	0.000				
1	0.017	0.999	0.017	46	0.719	0.695	1.04
2	0.035	0.999	0.035	47	0.731	0.682	1.07
3	0.052	0.999	0.052	48	0.743	0.669	1.11
4	0.070	0.998	0.070	49	0.755	0.656	1.15
5	0.087	0.996	0.087	50	0.766	0.643	1.19
6	0.105	0.995	0.105	51	0.777	0.629	1.23
7	0.122	0.993	0.123	52	0.788	0.616	1.28
8	0.139	0.990	0.141	53	0.799	0.602	1.33
9	0.156	0.988	0.158	54	0.809	0.588	1.38
10	0.174	0.985	0.176	55	0.819	0.574	1.43
11	0.191	0.982	0.194	56	0.829	0.559	1.48
12	0.208	0.978	0.213	57	0.839	0.545	1.54
13	0.225	0.974	0.231	58	0.848	0.530	1.60
14	0.242	0.970	0.249	59	0.857	0.515	1.66
15	0.259	0.966	0.268	60	0.866	0.500	1.73
16	0.276	0.961	0.287	61	0.875	0.485	1.80
17	0.292	0.956	0.306	62	0.883	0.469	1.88
18	0.309	0.951	0.325	63	0.891	0.454	1.96
19	0.326	0.946	0.344	64	0.989	0.438	2.05
20	0.342	0.940	0.364	65	0.906	0.423	2.14
21	0.358	0.934	0.384	66	0.914	0.407	2.25
22	0.375	0.927	0.404	67	0.921	0.391	2.36
23	0.391	0.921	0.424	68	0.927	0.375	2.48
24	0.407	0.914	0.445	69	0.934	0.358	2.61
25	0.423	0.906	0.466	70	0.940	0.342	2.75
26	0.438	0.898	0.488	71	0.946	0.326	2.90
27	0.454	0.891	0.510	72	0.951	0.309	3.08
28	0.469	0.883	0.532	73	0.956	0.292	3.27
29	0.485	0.875	0.554	74	0.961	0.276	3.49
30	0.500	0.866	0.577	75	0.966	0.259	3.73
31	0.515	0.857	0.601	76	0.970	0.242	4.01
32	0.530	0.848	0.625	77	0.974	0.225	4.33
33	0.545	0.839	0.649	78	0.978	0.208	4.70
34	0.559	0.829	0.675	79	0.982	0.191	5.14
35	0.574	0.819	0.700	80	0.985	0.174	5.67
36	0.588	0.809	0.727	81	0.988	0.156	6.31
37	0.602	0.799	0.754	82	0.990	0.139	7.12
38	0.616	0.788	0.781	83	0.993	0.122	8.14
39	0.629	0.777	0.810	84	0.995	0.105	9.51
40	0.643	0.766	0.839	85	0.996	0.087	11.43
41	0.656	0.755	0.869	86	0.998	0.070	14.30
42	0.669	0.743	0.900	87	0.999	0.052	19.08
43	0.682	0.731	0.933	88	0.999	0.035	28.64
44	0.695	0.719	0.966	89	0.999	0.017	57.28
45	0.707	0.707	1.000	90	1.000	0.000	Infinity

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